

Certain expansion formulae of incomplete I-functions associated with the Leibniz rule

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Abstract

In this paper, we determine some expansion formulae of the incomplete I-functions in affiliation with the Leibniz rule for the Riemann-Liouville type derivatives. Further, expansion formulae of the incomplete $\bar{I}$ -function, incomplete $\bar{H}$ -function and incomplete H-function are conferred as extraordinary instances of our primary outcomes.

Keywords: Incomplete Gamma functions, Incomplete I-functions, Fractional calculus, R-L fractional integral and derivative, Leibniz rule.

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1. Introduction

Rathie [1] described I-function in the following way:

$$I_{p,q}^{m,n}[z] = I_{p,q}^{m,n} \left[z \left| \begin{matrix} (\epsilon_1, \nu_1; \mathcal{E}_1), \dots, (\epsilon_p, \nu_p; \mathcal{E}_p) \\ (f_1, \omega_1; \mathcal{F}_1), \dots, (f_p, \omega_p; \mathcal{F}_p) \end{matrix} \right. \right] \\ = \frac{1}{2\pi i} \int_{\mathcal{L}} \psi(s) z^s ds, \quad (1.1)$$

where,

$$\psi(s) = \frac{\prod_{j=1}^m [\Gamma(f_j - \omega_j s)]^{\mathcal{F}_j} \prod_{j=1}^n [\Gamma(1 - \epsilon_j + \nu_j s)]^{\mathcal{E}_j}}{\prod_{j=m+1}^q [\Gamma(1 - f_j + \omega_j s)]^{\mathcal{F}_j} \prod_{j=n+1}^p [\Gamma(\epsilon_j - \nu_j s)]^{\mathcal{E}_j}}, \quad (1.2)$$

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and $m, n, p, q \in \mathbb{N}_0$ with $0 \leq n \leq p$, $0 \leq m \leq q$, $\nu_j, \varepsilon_j (j = 1, \dots, p)$, $\omega_j, \mathcal{F}_j (j = 1, \dots, q) \in \mathbb{R}^+$, $\epsilon_j, f_j \in \mathbb{C}$. The suitable conditions for the $\mathcal{L}$ contour convergence depicted in (1.1) and portrayals just as the I-function information can be found in [1].

We next characterized the recognizable lower and upper incomplete gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$, separately as:

$$\gamma(s, x) := \int_0^x y^{s-1} e^{-y} dy, \quad (\Re(s) > 0; x \geq 0) \quad (1.3)$$

and

$$\Gamma(s, x) := \int_x^\infty y^{s-1} e^{-y} dy, \quad (x \geq 0; \Re(s) > 0 \text{ if } x = 0). \quad (1.4)$$

These functions satisfy the accompanying deterioration connection :

$$\gamma(s, x) + \Gamma(s, x) = \Gamma(s) \quad (\Re(s) > 0). \quad (1.5)$$

Srivastava et al. [2] discovered a pair of Mellin-Barnes contour integral form of incomplete H-functions $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ and incomplete $\bar{H}$ -functions $\bar{\gamma}_{p,q}^{m,n}(z)$ and $\bar{\Gamma}_{p,q}^{m,n}(z)$ in terms of the $\gamma(s, x)$ and $\Gamma(s, x)$ represented by 1.3 and 1.4, respectively,

$$\gamma_{p,q}^{m,n}(z) = \gamma_{p,q}^{m,n} \left[z \left| \begin{matrix} (\epsilon_1, \nu_1, x), (\epsilon_j, \nu_j)_{2,p} \\ (f_j, \omega_j)_{1,q} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} g(s, x) z^{-s} ds \quad (1.6)$$

and

$$\Gamma_{p,q}^{m,n}(z) = \Gamma_{p,q}^{m,n} \left[z \left| \begin{matrix} (\epsilon_1, \nu_1, x), (\epsilon_j, \nu_j)_{2,p} \\ (f_j, \omega_j)_{1,q} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} G(s, x) z^{-s} ds, \quad (1.7)$$

where,

$$g(s, x) = \frac{\gamma(1 - \epsilon_1 - \nu_1 s, x) \prod_{j=1}^m \Gamma(f_j + \omega_j s) \prod_{j=2}^n \Gamma(1 - \epsilon_j - \nu_j s)}{\prod_{j=m+1}^q \Gamma(1 - f_j - \omega_j s) \prod_{j=n+1}^p \Gamma(\epsilon_j + \nu_j s)}, \quad (1.8)$$

and

$$G(s, x) = \frac{\Gamma(1 - \epsilon_1 - \nu_1 s, x) \prod_{j=1}^m \Gamma(f_j + \omega_j s) \prod_{j=2}^n \Gamma(1 - \epsilon_j - \nu_j s)}{\prod_{j=m+1}^q \Gamma(1 - f_j - \omega_j s) \prod_{j=n+1}^p \Gamma(\epsilon_j + \nu_j s)}, \quad (1.9)$$

with the arrangement of conditions set out in [2].

These incomplete H-functions satisfy the accompanying deterioration connection:

$$\gamma_{p,q}^{m,n}(z) + \Gamma_{p,q}^{m,n}(z) = H_{p,q}^{m,n}(z). \quad (1.10)$$

The incomplete H-functions $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ described in (1.6) and (1.7) exist for $x \geq 0$, under the set of conditions given by Srivastava et al. [2].

Jangid et al. [3] presented a group of the incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}(z)$ and $\Gamma I_{p,q}^{m,n}(z)$ which prompts a characteristic speculation and disintegration equation for I-function:

$$\begin{aligned} {}^{\gamma}I_{p,q}^{m,n}[z] &= {}^{\gamma}I_{p,q}^{m,n} \left[z \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : \chi), (\epsilon_2, \nu_2; \mathcal{E}_2), \dots, (\epsilon_p, \nu_p; \mathcal{E}_p) \\ (f_1, \omega_1; \mathcal{F}_1), (f_2, \omega_2; \mathcal{F}_2), \dots, (f_q, \omega_q; \mathcal{F}_q) \end{array} \right. \right] \\ &= \frac{1}{2\pi i} \int_{\mathcal{L}} \phi(s, \chi) z^s ds, \end{aligned} \quad (1.11)$$

and

$$\begin{aligned} \Gamma I_{p,q}^{m,n}[z] &= \Gamma I_{p,q}^{m,n} \left[z \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : \chi), (\epsilon_2, \nu_2; \mathcal{E}_2), \dots, (\epsilon_p, \nu_p; \mathcal{E}_p) \\ (f_1, \omega_1; \mathcal{F}_1), (f_2, \omega_2; \mathcal{F}_2), \dots, (f_q, \omega_q; \mathcal{F}_q) \end{array} \right. \right] \\ &= \frac{1}{2\pi i} \int_{\mathcal{L}} \Phi(s, \chi) z^s ds, \end{aligned} \quad (1.12)$$

for all $z \neq 0$, where,

$$\phi(s, \chi) = \frac{[\gamma(1 - \epsilon_1 + \nu_1 s, \chi)]^{\mathcal{E}_1} \prod_{j=1}^m [\Gamma(f_j - \omega_j s)]^{\mathcal{F}_j} \prod_{j=2}^n [\Gamma(1 - \epsilon_j + \nu_j s)]^{\mathcal{E}_j}}{\prod_{j=m+1}^q [\Gamma(1 - f_j + \omega_j s)]^{\mathcal{F}_j} \prod_{j=n+1}^p [\Gamma(\epsilon_j - \nu_j s)]^{\mathcal{E}_j}}, \quad (1.13)$$

and

$$\Phi(s, \chi) = \frac{[\Gamma(1 - \epsilon_1 + \nu_1 s, \chi)]^{\mathcal{E}_1} \prod_{j=1}^m [\Gamma(f_j - \omega_j s)]^{\mathcal{F}_j} \prod_{j=2}^n [\Gamma(1 - \epsilon_j + \nu_j s)]^{\mathcal{E}_j}}{\prod_{j=m+1}^q [\Gamma(1 - f_j + \omega_j s)]^{\mathcal{F}_j} \prod_{j=n+1}^p [\Gamma(\epsilon_j - \nu_j s)]^{\mathcal{E}_j}}. \quad (1.14)$$

The incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}[z]$ and $\Gamma I_{p,q}^{m,n}[z]$ in 1.11 and 1.12 exist for all $\chi \geq 0$ under indistinguishable form and conditions from expressed in Rathie [1]. For $\mathcal{E}_1 = 1$, the definitions 1.11 and 1.12 at once yield the following division relation:

$${}^{\gamma}I_{p,q}^{m,n}[z] + \Gamma I_{p,q}^{m,n}[z] = I_{p,q}^{m,n}[z] \quad (\mathcal{E}_1 = 1), \quad (1.15)$$

for the familiar I-function.

The incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}[z]$ and $\Gamma I_{p,q}^{m,n}[z]$ defined in 1.11 and 1.12 exist for all $\chi \geq 0$, under the set of conditions given by Rathie [1], with

$$\Delta > 0, |\arg z| < \Delta \frac{\pi}{2},$$

where,

$$\Delta = \sum_{j=1}^m \mathcal{F}_j \omega_j - \sum_{j=m+1}^q \mathcal{F}_j \omega_j + \sum_{j=1}^n \mathcal{E}_j \nu_j - \sum_{j=n+1}^p \mathcal{E}_j \nu_j.$$

Numerous authors are currently working on variety applications of these incomplete functions. Recently, Bansal et al. [4, 5, 6, 7] discussed about some of the new outcomes

and uses of the incomplete H-functions; such as they solved the Fredholm-type integral equation involving incomplete H-function and incomplete $\bar{H}$ -function in the kernel, useful classical integral transforms of the $\aleph$ -functions, some new outcomes of Srivastava-Luo-Raina $\mathbb{M}$ -transform associated to incomplete H-function, and Oberhettinger's integral formula involving M-series. Recent studies [8, 9, 10, 11] for further details on the characteristics of incomplete I-functions.

In [12, 13, 14], the authors discussed an extension of the generalized Mittag-leffler function in terms of the generalized Pochhammer symbol, determined some results of the generalized Bessel-Maitland function in the field of the fractional calculus and new extension of Caputo fractional derivative operator.

The Riemann-Liouville fractional operators of order ν for $f(z)$ are characterized as follows (see [15]):

$$I^\nu f(z) = \frac{1}{\Gamma(\nu)} \int_0^z (z-t)^{\nu-1} f(t) dt, \quad (1.16)$$

here, the integral is all around characterized gave f is a locally integrable function and ν is a complex number in the half plane $\Re(\nu) > 0$.

$$D_z^\nu f(z) = \frac{1}{\Gamma(-\nu)} \int_0^z (z-t)^{(-\nu-1)} f(t) dt \quad (\Re(\nu) < 0), \quad (1.17)$$

If $\Re(\nu) \geq 0$ and $m \in \mathbb{N}$ is the smallest integer with $m-1 \leq \Re(\nu) < m$, then

$$D_z^\nu f(z) = \frac{d^m}{dz^m} D_z^{\nu-m} f(z) = \frac{d^m}{dz^m} \left[\frac{1}{\Gamma(-\nu+m)} \int_0^z (z-t)^{-\nu+m-1} f(t) dt \right]. \quad (1.18)$$

The classical Leibniz rule for two differentiable functions f and g is defined as follows:

$$D^n[f(t)g(t)] = \sum_{k=0}^n \binom{n}{k} [D^k g(t)][D^{n-k} f(t)].$$

This Leibniz rule can be reached out for the Riemann-Liouville type derivatives. On the off chance that f and g are two functions of class C , at that point the fractional speculation of the Leibniz rule is characterized as (see [16])

$$D^\mu[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{\mu}{k} [D^k g(t)][D^{\mu-k} f(t)]; \quad \mu > 0, \quad k \in \mathbb{N}. \quad (1.19)$$

In particular, if f is function of class C , then

$$D^\mu[t^p f(t)] = \sum_{r=0}^p \binom{\mu}{r} [D^r t^p][D^{\mu-r} f(t)], \quad \mu > 0.$$

The Leibniz rule, which sums up the differentiation law of the product, might be utilized to extricate an instrument that ascertains the organization portrayal of the differential operators.

2. Main Results

We establish some expansion formulae of the incomplete I-functions by the use of the Leibniz rule.

Theorem 2.1. *Let $\lambda \geq 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m - 1 \leq \Re(\mu) \leq m$, then the following result holds:*

$$\begin{aligned} & \Gamma_{p+1, q+1}^{m, n+1} \left[az^\sigma \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (1 - \lambda, \sigma; 1), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p} \\ (f_j, \omega_j; \mathcal{F}_j)_{1,q}, (1 - \lambda + \mu, \sigma; 1) \end{array} \right. \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \Gamma_{p+1, q+1}^{m, n+1} \left[az^\sigma \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (0, \sigma; 1), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p} \\ (f_j, \omega_j; \mathcal{F}_j)_{1,q}, (k, \sigma; 1) \end{array} \right. \right]. \end{aligned} \quad (2.1)$$

Proof. To prove the result 2.1, let us consider $f(z) = z^{\lambda-1}$ and

$$g(z) = \Gamma_{p,q}^{m,n}[az^\sigma] = \Gamma_{p,q}^{m,n} \left[az^\sigma \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p} \\ (f_j, \omega_j; \mathcal{F}_j)_{1,q} \end{array} \right. \right].$$

Now substituting the values of $f(z)$ and $g(z)$ in 1.19, we obtain

$$D^\mu [z^{\lambda-1} \Gamma_{p,q}^{m,n}(az^\sigma)] = \sum_{k=0}^{\infty} \binom{\mu}{k} [D^k \Gamma_{p,q}^{m,n}(az^\sigma)] [D^{\mu-k} z^{\lambda-1}]. \quad (2.2)$$

On taking L.H.S of equation 2.2, we obtain

$$\begin{aligned} D^\mu [z^{\lambda-1} \Gamma_{p,q}^{m,n}(az^\sigma)] &= D^\mu \left[z^{\lambda-1} \frac{1}{2\pi i} \int_{\mathcal{L}} \Phi(s, x) a^s z^{\sigma s} ds \right] \\ &= \frac{1}{2\pi i} \int_{\mathcal{L}} \Phi(s, x) a^s D^\mu [z^{\lambda+\sigma s-1}] ds \\ &= \frac{1}{2\pi i} \int_{\mathcal{L}} \Phi(s, x) a^{-s} \frac{\Gamma(\lambda + \sigma s)}{\Gamma(\lambda - \mu + \sigma s)} z^{\lambda-\mu+\sigma s-1} ds. \end{aligned}$$

In view of the definition 1.12, we have

$$\begin{aligned} D^\mu [z^{\lambda-1} \Gamma_{p,q}^{m,n}(az^\sigma)] &= z^{\lambda-\mu-1} \times \\ &\quad \Gamma_{p+1, q+1}^{m, n+1} \left[az^\sigma \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (1 - \lambda, \sigma; 1), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p} \\ (f_j, \omega_j; \mathcal{F}_j)_{1,q}, (1 - \lambda + \mu, \sigma; 1) \end{array} \right. \right]. \end{aligned} \quad (2.3)$$

Similarly, the R.H.S. of equation 2.2 is the immediate consequences of the definitions 1.12 and 1.19, we obtain

$$\begin{aligned} & \sum_{k=0}^{\infty} \binom{\mu}{k} [D^k \Gamma_{p,q}^{m,n}(az^\sigma)] [D^{\mu-k} z^{\lambda-1}] \\ &= z^{\lambda-\mu-1} \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \Gamma_{p+1, q+1}^{m, n+1} \left[az^\sigma \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (0, \sigma), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p} \\ (f_j, \omega_j; \mathcal{F}_j)_{1,q}, (k, \sigma; 1) \end{array} \right. \right]. \end{aligned} \quad (2.4)$$

Substituting the equation 2.3 and 2.4 into 2.2, we get the required result 2.1. $\square$

Below theorem are the immediate consequences of the definitions 1.11, 1.12 and 1.19 and hence they are given without proof here.

Theorem 2.2. Let $\lambda \geq 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:

$$\begin{aligned} & \Gamma I_{p+1, q+1}^{m, n+1} \left[az^{-\sigma} \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p}, (\lambda - \mu, \sigma; 1) \\ (\lambda, \sigma; 1), (f_j, \omega_j; \mathcal{F}_j)_{1,q} \end{array} \right. \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \Gamma I_{p+1, q+1}^{m, n+1} \left[az^{-\sigma} \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p}, (1 - k, \sigma; 1) \\ (1, \sigma; 1), (f_j, \omega_j; \mathcal{F}_j)_{1,q} \end{array} \right. \right]. \end{aligned} \quad (2.5)$$

Theorem 2.3. Let $\lambda \geq 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:

$$\begin{aligned} & \gamma I_{p+1, q+1}^{m, n+1} \left[az^{\sigma} \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (1 - \lambda, \sigma; 1), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p} \\ (f_j, \omega_j; \mathcal{F}_j)_{1,q}, (1 - \lambda + \mu, \sigma; 1) \end{array} \right. \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \gamma I_{p+1, q+1}^{m, n+1} \left[az^{\sigma} \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (0, \sigma; 1), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p} \\ (f_j, \omega_j; \mathcal{F}_j)_{1,q}, (k, \sigma; 1) \end{array} \right. \right]. \end{aligned} \quad (2.6)$$

Theorem 2.4. Let $\lambda \geq 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:

$$\begin{aligned} & \gamma I_{p+1, q+1}^{m, n+1} \left[az^{-\sigma} \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p}, (\lambda - \mu, \sigma; 1) \\ (\lambda, \sigma; 1), (f_j, \omega_j; \mathcal{F}_j)_{1,q} \end{array} \right. \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \gamma I_{p+1, q+1}^{m, n+1} \left[az^{-\sigma} \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p}, (1 - k, \sigma; 1), \\ (1, \sigma; 1), (f_j, \omega_j; \mathcal{F}_j)_{1,q} \end{array} \right. \right]. \end{aligned} \quad (2.7)$$

3. Special Cases

In this section, we determine a few expansion formulae for the incomplete $\bar{I}$ -function, incomplete $\bar{H}$ -function and incomplete H -function, as special instances of Theorem 2.1 and Theorem 2.2. For the meanings of incomplete $\bar{H}$ -function and incomplete H -function one could see Srivastava [2]. To delineate the uses of fundamental outcomes, if we give the specific qualities to the parameters, then we obtain the following special cases:

(i) Incomplete $\bar{I}$ -function: If we set $\mathcal{F}_j = 1$ ($j = 1, \dots, m$) in 1.12 and utilizing the connection, that is

$$\Gamma \bar{I}_{p, q}^{m, n}(z) = \Gamma I_{p, q}^{m, n} \left[z \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p} \\ (f_j, \omega_j; 1)_{1,m}, (f_j, \omega_j, \mathcal{F}_j)_{m+1,q} \end{array} \right. \right], \quad (3.1)$$

in 2.1 and 2.5, respectively, at that point we get the accompanying corollaries:

Corollary 3.1. Let $\lambda \geq 1$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following formula holds:

$$\begin{aligned} & \Gamma_{\bar{I}_{p+1, q+1}}^{m, n+1} \left[az^\sigma \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (1-\lambda, \sigma; 1), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p} \\ (f_j, \omega_j; 1)_{1,m}, (f_j, \omega_j; \mathcal{F}_j)_{m+1,q}, (1-\lambda+\mu, \sigma; 1) \end{array} \right. \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)} \Gamma_{\bar{I}_{p+1, q+1}}^{m, n+1} \left[az^\sigma \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (0, \sigma; 1), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p} \\ (f_j, \omega_j; 1)_{1,m}, (f_j, \omega_j; \mathcal{F}_j)_{m+1,q}, (k, \sigma; 1) \end{array} \right. \right]. \end{aligned} \quad (3.2)$$

Corollary 3.2. Let $\lambda \geq 1$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:

$$\begin{aligned} & \Gamma_{\bar{I}_{p+1, q+1}}^{m, n+1} \left[az^{-\sigma} \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p}, (\lambda-\mu, \sigma; 1) \\ (\lambda, \sigma; 1), (f_j, \omega_j; 1)_{1,m}, (f_j, \omega_j; \mathcal{F}_j)_{m+1,q} \end{array} \right. \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)} \Gamma_{\bar{I}_{p+1, q+1}}^{m, n+1} \left[az^{-\sigma} \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,p}, (1-k, \sigma; 1) \\ (1, \sigma; 1), (f_j, \omega_j; 1)_{1,m}, (f_j, \omega_j; \mathcal{F}_j)_{m+1,q} \end{array} \right. \right]. \end{aligned} \quad (3.3)$$

(ii) **Incomplete $\bar{H}$ -function $\bar{\Gamma}_{p,q}^{m,n}(z)$:** if we give particular values to the parameters, such as $\mathcal{F}_j$ ($j = 1, \dots, m$) = 1 and $\mathcal{E}_j$ ($j = n+1, \dots, p$) = 1 in 1.12, and using the below relation (see [2])

$$\begin{aligned} \bar{\Gamma}_{p,q}^{m,n}(z) &= \Gamma_{\bar{I}_{p,q}}^{m,n} \left[z \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,n}, (\epsilon_j, \nu_j; 1)_{n+1,p} \\ (f_j, \omega_j; 1)_{1,m}, (f_j, \omega_j; \mathcal{F}_j)_{m+1,q} \end{array} \right. \right] \\ &= \bar{\Gamma}_{p,q}^{m,n} \left[z \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,n}, (\epsilon_j, \nu_j)_{n+1,p} \\ (f_j, \omega_j)_{1,m}, (f_j, \omega_j; \mathcal{F}_j)_{m+1,q} \end{array} \right. \right], \end{aligned} \quad (3.4)$$

we get the following corollaries:

Corollary 3.3. Let $\lambda \geq 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:

$$\begin{aligned} & \bar{\Gamma}_{p+1, q+1}^{m, n+1} \left[az^\sigma \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (1-\lambda, \sigma; 1), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,n}, \\ (f_j, \omega_j)_{1,m}, (f_j, \omega_j; \mathcal{F}_j)_{m+1,q}, (1-\lambda+\mu, \sigma; 1) \end{array} \right. \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)} \times \\ & \quad \bar{\Gamma}_{p+1, q+1}^{m, n+1} \left[az^\sigma \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (0, \sigma; 1), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2,n}, (\epsilon_j, \nu_j)_{n+1,p} \\ (f_j, \omega_j)_{1,m}, (f_j, \omega_j; \mathcal{F}_j)_{m+1,q}, (k, \sigma; 1) \end{array} \right. \right]. \end{aligned} \quad (3.5)$$

Corollary 3.4. Let $\lambda \geq 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leq \Re(\mu) \leq m$, then the following result holds:

$$\begin{aligned} & \bar{\Gamma}_{p+1, q+1}^{m, n+1} \left[az^{-\sigma} \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2, n}, (\epsilon_j, \nu_j)_{n+1, p}, (\lambda - \mu, \sigma) \\ (\lambda, \sigma), (f_j, \omega_j)_{1, m}, (f_j, \omega_j; \mathcal{F}_j)_{m+1, q} \end{array} \right. \right] \\ &= \sum_{k=0}^{\infty} \binom{\mu}{k} \frac{\Gamma(\lambda)}{\Gamma(\lambda - \mu + k)} \times \\ & \bar{\Gamma}_{p+1, q+1}^{m, n+1} \left[az^{-\sigma} \left| \begin{array}{c} (\epsilon_1, \nu_1; \mathcal{E}_1 : x), (\epsilon_j, \nu_j; \mathcal{E}_j)_{2, n}, (\epsilon_j, \nu_j)_{n+1, p}, (1 - k, \sigma) \\ (1, \sigma), (f_j, \omega_j)_{1, m}, (f_j, \omega_j; \mathcal{F}_j)_{m+1, q} \end{array} \right. \right]. \quad (3.6) \end{aligned}$$

(iii) **Incomplete H-function** $\Gamma_{p, q}^{m, n}(z)$: if we put $\mathcal{F}_j$ ($j = 1, \dots, q$) = 1 and $\mathcal{E}_j$ ($j = 1, \dots, p$) = 1 in 1.12, and utilizing the connection, that is (see [2])

$$\begin{aligned} \Gamma_{p, q}^{m, n}(z) &= \Gamma_{p, q}^{m, n} \left[z \left| \begin{array}{c} (\epsilon_1, \nu_1; 1 : x), (\epsilon_j, \nu_j; 1)_{2, p} \\ (f_j, \omega_j; 1)_{1, q} \end{array} \right. \right] \\ &= \Gamma_{p, q}^{m, n} \left[z \left| \begin{array}{c} (\epsilon_1, \nu_1 : x), (\epsilon_j, \nu_j)_{2, p} \\ (f_j, \omega_j)_{1, q} \end{array} \right. \right], \quad (3.7) \end{aligned}$$

in 2.1 and 2.5 respectively, then we get the results obtained by Meena et al. [17, Theorem 2.1. and Theorem 2.2.].

Remark: Similarly, special cases for the Theorem 2.3 and Theorem 2.4 may be derived.

4. Conclusion

In this paper, by making use of the Leibniz rule for the Riemann-Liouville type derivatives, we have obtained certain expansion formulae of the incomplete I-functions. For our main findings, we have discussed some special cases and some known results also appeared.

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