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Certain expansion formulae of incomplete I-functions associated with the Leibniz rule

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Abstract

In this paper, we determine some expansion formulae of the incomplete I-functions in affiliation with the Leibniz rule for the Riemann-Liouville type derivatives. Further, expansion formulae of the incomplete $\bar{I}$-function, incomplete $\overline{H}$-function and incomplete H-function are conferred as extraordinary instances of our primary outcomes.

Keywords: Incomplete Gamma functions, Incomplete I-functions, Fractional calculus, R-L fractional integral and derivative, Leibniz rule.
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1. Introduction

Rathie [1] described I-function in the following way:

$$I_{p,q}^{m,n}[z] = I_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (\mathfrak{e}_1, \nu_1; \mathcal{E}_1), \ldots, (\mathfrak{e}_p, \nu_p; \mathcal{E}_p) \\ (\mathfrak{f}_1, \omega_1; \mathcal{F}_1), \ldots, (\mathfrak{f}_p, \omega_p; \mathcal{F}_p) \end{matrix} \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \psi(s)\, z^s \, ds, \tag{1.1}$$

where,

$$\psi(s) = \frac{\prod_{j=1}^{m} [\Gamma(\mathfrak{f}_j - \omega_j s)]^{\mathcal{F}_j} \prod_{j=1}^{n} [\Gamma(1 - \mathfrak{e}_j + \nu_j s)]^{\mathcal{E}_j}}{\prod_{j=m+1}^{q} [\Gamma(1 - \mathfrak{f}_j + \omega_j s)]^{\mathcal{F}_j} \prod_{j=n+1}^{p} [\Gamma(\mathfrak{e}_j - \nu_j s)]^{\mathcal{E}_j}}, \tag{1.2}$$

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and $m, n, p, q \in \mathbb{N}_0$ with $0 \leqslant n \leqslant p$, $0 \leqslant m \leqslant q$, $\nu_j, \mathcal{E}_j (j = 1, .., p)$, $\omega_j, \mathcal{F}_j (j = 1, ..., q) \in \mathbb{R}^+$, $\mathfrak{e}_j, \mathfrak{f}_j \in \mathbb{C}$. The suitable conditions for the $\mathcal{L}$ contour convergence depicted in (1.1) and portrayals just as the I-function information can be found in [1].

We next characterized the recognizable lower and upper incomplete gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$, separately as:

$$\gamma(s, x) := \int_0^x y^{s-1} \, e^{-y} \, dy, \qquad \left(\Re(s) > 0; \; x \geqq 0\right) \tag{1.3}$$

and

$$\Gamma(s, x) := \int_x^\infty y^{s-1} \, e^{-y} \, dy, \qquad \left(x \geqq 0; \; \Re(s) > 0 \quad \text{if} \quad x = 0\right). \tag{1.4}$$

These functions satisfy the accompanying deterioration connection :

$$\gamma(s, x) + \Gamma(s, x) = \Gamma(s) \;\; (\Re(s) > 0). \tag{1.5}$$

Srivastava et al. [2] discovered a pair of Mellin-Barnes contour integral form of incomplete H-functions $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ and incomplete $\overline{H}$-functions $\overline{\gamma}_{p,q}^{m,n}(z)$ and $\overline{\Gamma}_{p,q}^{m,n}(z)$ in terms of the $\gamma(s, x)$ and $\Gamma(s, x)$ represented by 1.3 and 1.4, respectively,

$$\gamma_{p,\,q}^{m,\,n}(z) = \gamma_{p,\,q}^{m,\,n}\left[z \left| \begin{array}{c} (\mathfrak{e}_1, \nu_1, x), (\mathfrak{e}_j, \nu_j)_{2,p} \\ (\mathfrak{f}_j, \omega_j)_{1,q} \end{array} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} g(s, x) \, z^{-s} \, ds \tag{1.6}$$

and

$$\Gamma_{p,\,q}^{m,\,n}(z) = \Gamma_{p,\,q}^{m,\,n}\left[z \left| \begin{array}{c} (\mathfrak{e}_1, \nu_1, x), (\mathfrak{e}_j, \nu_j)_{2,p} \\ (\mathfrak{f}_j, \omega_j)_{1,q} \end{array} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} G(s, x) \, z^{-s} \, ds, \tag{1.7}$$

where,

$$g(s, x) = \frac{\gamma(1 - \mathfrak{e}_1 - \nu_1 s, x) \prod_{j=1}^{m} \Gamma(\mathfrak{f}_j + \omega_j s) \prod_{j=2}^{n} \Gamma(1 - \mathfrak{e}_j - \nu_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - \mathfrak{f}_j - \omega_j s) \prod_{j=n+1}^{p} \Gamma(\mathfrak{e}_j + \nu_j s)}, \tag{1.8}$$

and

$$G(s, x) = \frac{\Gamma(1 - \mathfrak{e}_1 - \nu_1 s, x) \prod_{j=1}^{m} \Gamma(\mathfrak{f}_j + \omega_j s) \prod_{j=2}^{n} \Gamma(1 - \mathfrak{e}_j - \nu_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - \mathfrak{f}_j - \omega_j s) \prod_{j=n+1}^{p} \Gamma(\mathfrak{e}_j + \nu_j s)}, \tag{1.9}$$

with the arrangement of conditions setout in [2].

These incomplete H-functions satisfy the accompanying deterioration connection:

$$\gamma_{p,q}^{m,n}(z) + \Gamma_{p,q}^{m,n}(z) = H_{p,q}^{m,n}(z). \tag{1.10}$$

The incomplete H-functions $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ described in (1.6) and (1.7) exist for $x \geqslant 0$, under the set of conditions given by Srivastava et al. [2].

Jangid et al. [3] presented a group of the incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}(z)$ and ${}^{\Gamma}I_{p,q}^{m,n}(z)$ which prompts a characteristic speculation and disintegration equation for I-function:

$$
\begin{aligned}
{}^{\gamma}I_{p,q}^{m,n}[z] &= {}^{\gamma}I_{p,\,q}^{m,\,n}\left[z \,\middle|\, \begin{matrix} (\mathfrak{e}_1, \nu_1; \mathcal{E}_1 : x), (\mathfrak{e}_2, \nu_2; \mathcal{E}_2), ..., (\mathfrak{e}_p, \nu_p; \mathcal{E}_p) \\ (\mathfrak{f}_1, \omega_1; \mathcal{F}_1), (\mathfrak{f}_2, \omega_2; \mathcal{F}_2), ..., (\mathfrak{f}_q, \omega_q; \mathcal{F}_q) \end{matrix} \right] \\
&= \frac{1}{2\pi i} \int_{\mathcal{L}} \phi(s, x)\, z^s\, ds, \qquad (1.11)
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{\Gamma}I_{p,q}^{m,n}[z] &= {}^{\Gamma}I_{p,\,q}^{m,\,n}\left[z \,\middle|\, \begin{matrix} (\mathfrak{e}_1, \nu_1; \mathcal{E}_1 : x), (\mathfrak{e}_2, \nu_2; \mathcal{E}_2), ..., (\mathfrak{e}_p, \nu_p; \mathcal{E}_p) \\ (\mathfrak{f}_1, \omega_1; \mathcal{F}_1), (\mathfrak{f}_2, \omega_2; \mathcal{F}_2), ..., (\mathfrak{f}_q, \omega_q; \mathcal{F}_q) \end{matrix} \right] \\
&= \frac{1}{2\pi i} \int_{\mathcal{L}} \Phi(s, x)\, z^s\, ds, \qquad (1.12)
\end{aligned}
$$

for all $z \neq 0$, where,

$$
\phi(s, x) = \frac{[\gamma(1 - \mathfrak{e}_1 + \nu_1 s, x)]^{\mathcal{E}_1} \prod_{j=1}^{m} [\Gamma(\mathfrak{f}_j - \omega_j s)]^{\mathcal{F}_j} \prod_{j=2}^{n} [\Gamma(1 - \mathfrak{e}_j + \nu_j s)]^{\mathcal{E}_j}}{\prod_{j=m+1}^{q} [\Gamma(1 - \mathfrak{f}_j + \omega_j s)]^{\mathcal{F}_j} \prod_{j=n+1}^{p} [\Gamma(\mathfrak{e}_j - \nu_j s)]^{\mathcal{E}_j}}, \qquad (1.13)
$$

and

$$
\Phi(s, x) = \frac{[\Gamma(1 - \mathfrak{e}_1 + \nu_1 s, x)]^{\mathcal{E}_1} \prod_{j=1}^{m} [\Gamma(\mathfrak{f}_j - \omega_j s)]^{\mathcal{F}_j} \prod_{j=2}^{n} [\Gamma(1 - \mathfrak{e}_j + \nu_j s)]^{\mathcal{E}_j}}{\prod_{j=m+1}^{q} [\Gamma(1 - \mathfrak{f}_j + \omega_j s)]^{\mathcal{F}_j} \prod_{j=n+1}^{p} [\Gamma(\mathfrak{e}_j - \nu_j s)]^{\mathcal{E}_j}}. \qquad (1.14)
$$

The incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}[z]$ and ${}^{\Gamma}I_{p,q}^{m,n}[z]$ in 1.11 and 1.12 exist for all $x \geqslant 0$ under indistinguishable form and conditions from expressed in Rathie [1]. For $\mathcal{E}_1 = 1$, the definitions 1.11 and 1.12 at once yield the following division relation:

$$
{}^{\gamma}I_{p,q}^{m,n}[z] + {}^{\Gamma}I_{p,q}^{m,n}[z] = I_{p,q}^{m,n}[z] \quad (\mathcal{E}_1 = 1), \qquad (1.15)
$$

for the familiar I-function.
The incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}[z]$ and ${}^{\Gamma}I_{p,q}^{m,n}[z]$ defined in 1.11 and 1.12 exist for all $x \geqslant 0$, under the set of conditions given by Rathie [1], with

$$
\Delta > 0, |\arg z| < \Delta \frac{\pi}{2},
$$

where,

$$
\Delta = \sum_{j=1}^{m} \mathcal{F}_j \omega_j - \sum_{j=m+1}^{q} \mathcal{F}_j \omega_j + \sum_{j=1}^{n} \mathcal{E}_j \nu_j - \sum_{j=n+1}^{p} \mathcal{E}_j \nu_j.
$$

Numerous authors are currently working on variety applications of these incomplete functions. Recently, Bansal et al. [4, 5, 6, 7] discussed about some of the new outcomes and uses of the incomplete H-functions; such as they solved the Fredholm-type integral equation involving incomplete H-function and incomplete $\overline{\text{H}}$-function in the kernel, useful classical integral transforms of the $\aleph$-functions, some new outcomes of Srivastava-Luo-Raina $\mathbb{M}$-transform associated to incomplete H-function, and Oberhettinger's integral formula involving M-series. Recent studies [8, 9, 10, 11] for further details on the characteristics of incomplete I-functions.

In [12, 13, 14], the authors discussed an extension of the generalized Mittag-leffler function in terms of the generalized Pochhammer symbol, determined some results of the generalized Bessel-Maitland function in the field of the fractional calculus and new extention of Caputo fractional derivative operator.

The Riemann-Liouville fractional operators of order ν for $f(z)$ are characterized as follows (see [15]):

$$I^{\nu} f(z) = \frac{1}{\Gamma(\nu)} \int_0^z (z-t)^{\nu-1} f(t)\,dt, \tag{1.16}$$

here, the integral is all around characterized gave f is a locally integrable function and ν is a complex number in the half plane $\Re(\nu) > 0$.

$$D_z^{\nu} f(z) = \frac{1}{\Gamma(-\nu)} \int_0^z (z-t)^{(-\nu-1)} f(t)\,dt \quad (\Re(\nu) < 0), \tag{1.17}$$

If $\Re(\nu) \geqslant 0$ and $m \in \mathbb{N}$ is the smallest integer with $m-1 \leqslant \Re(\nu) < m$, then

$$D_z^{\nu} f(z) = \frac{d^m}{dz^m} D_z^{\nu-m} f(z) == \frac{d^m}{dz^m}\left[\frac{1}{\Gamma(-\nu+m)} \int_0^z (z-t)^{-\nu+m-1} f(t)\,dt\right]. \tag{1.18}$$

The classical Leibniz rule for two differentiable functions f and g is defined as follows:

$$D^n[f(t)g(t)] = \sum_{k=0}^{n} \binom{n}{k} [D^k g(t)][D^{n-k} f(t)].$$

This Leibniz rule can be reached out for the Riemann-Liouville type derivatives. On the off chance that f and g are two functions of class C, at that point the fractional speculation of the Leibniz rule is characterized as (see [16])

$$D^{\mu}[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{\mu}{k} [D^k g(t)][D^{\mu-k} f(t)]\ ; \quad \mu > 0, \quad k \in \mathbb{N}. \tag{1.19}$$

In particular, if f is function of class C, then

$$D^{\mu}[t^p f(t)] = \sum_{r=0}^{p} \binom{\mu}{r} [D^r t^p][D^{\mu-r} f(t)], \qquad \mu > 0.$$

The Leibniz rule, which sums up the differentiation law of the product, might be utilized to extricate an instrument that ascertains the organization portrayal of the differential operators.

2. Main Results

We establish some expansion formulae of the incomplete I-functions by the use of the Leibniz rule.

Theorem 2.1. *Let $\lambda \geqslant 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $m-1 \leqslant \Re(\mu) \leqslant m$, then the following result holds:*

$$
{}^{\Gamma}I^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{array}{c}(e_1,\nu_1;\mathcal{E}_1:x),(1-\lambda,\sigma;1),(e_j,\nu_j;\mathcal{E}_j)_{2,p}\\(f_j,\omega_j;\mathcal{F}_j)_{1,q},(1-\lambda+\mu,\sigma;1)\end{array}\right]
= \sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\,{}^{\Gamma}I^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{array}{c}(e_1,\nu_1;\mathcal{E}_1:x),(0,\sigma;1),(e_j,\nu_j;\mathcal{E}_j)_{2,p}\\(f_j,\omega_j;\mathcal{F}_j)_{1,q},(k,\sigma;1)\end{array}\right]. \tag{2.1}
$$

Proof. To prove the result 2.1, let us consider $f(z) = z^{\lambda-1}$ and

$$
g(z) = {}^{\Gamma}I^{m,n}_{p,q}[az^{\sigma}] = {}^{\Gamma}I^{m,\,n}_{p,\,q}\left[az^{\sigma}\,\middle|\,\begin{array}{c}(e_1,\nu_1;\mathcal{E}_1:x),(e_j,\nu_j;\mathcal{E}_j)_{2,p}\\(f_j,\omega_j;\mathcal{F}_j)_{1,q}\end{array}\right].
$$

Now substituting the values of $f(z)$ and $g(z)$ in 1.19, we obtain

$$
D^{\mu}\left[z^{\lambda-1}\,{}^{\Gamma}I^{m,n}_{p,q}(az^{\sigma})\right] = \sum_{k=0}^{\infty}\binom{\mu}{k}\left[D^{k}\,{}^{\Gamma}I^{m,n}_{p,q}(az^{\sigma})\right]\left[D^{\mu-k}z^{\lambda-1}\right]. \tag{2.2}
$$

On taking L.H.S of equation 2.2, we obtain

$$
\begin{aligned}
D^{\mu}\left[z^{\lambda-1}\,{}^{\Gamma}I^{m,n}_{p,q}(az^{\sigma})\right] &= D^{\mu}\left[z^{\lambda-1}\frac{1}{2\pi i}\int_{\mathcal{L}}\Phi(s,x)a^{s}z^{\sigma s}\,ds\right]\\
&= \frac{1}{2\pi i}\int_{\mathcal{L}}\Phi(s,x)a^{s}D^{\mu}\left[z^{\lambda+\sigma s-1}\right]\,ds\\
&= \frac{1}{2\pi i}\int_{\mathcal{L}}\Phi(s,x)a^{-s}\frac{\Gamma(\lambda+\sigma s)}{\Gamma(\lambda-\mu+\sigma s)}z^{\lambda-\mu+\sigma s-1}\,ds.
\end{aligned}
$$

In view of the definition 1.12, we have

$$
D^{\mu}\left[z^{\lambda-1}\,{}^{\Gamma}I^{m,n}_{p,q}(az^{\sigma})\right] = z^{\lambda-\mu-1}\times
{}^{\Gamma}I^{m,n+1}_{p+1,q+1}\left[az^{\sigma}\,\middle|\,\begin{array}{c}(e_1,\nu_1;\mathcal{E}_1:x),(1-\lambda,\sigma;1),(e_j,\nu_j;\mathcal{E}_j)_{2,p}\\(f_j,\omega_j;\mathcal{F}_j)_{1,q}\,,(1-\lambda+\mu,\sigma;1)\end{array}\right]. \tag{2.3}
$$

Similarly, the R.H.S. of equation 2.2 is the immediate consequences of the definitions 1.12 and 1.19, we obtain

$$
\sum_{k=0}^{\infty}\binom{\mu}{k}\left[D^{k}\,{}^{\Gamma}I^{m,n}_{p,q}(az^{\sigma})\right]\left[D^{\mu-k}z^{\lambda-1}\right]
= z^{\lambda-\mu-1}\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\,{}^{\Gamma}I^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{array}{c}(e_1,\nu_1;\mathcal{E}_1:x),(0,\sigma),(e_j,\nu_j;\mathcal{E}_j)_{2,p}\\(f_j,\omega j;\mathcal{F}_j)_{1,q}\,,(k,\sigma;1)\end{array}\right]. \tag{2.4}
$$

Substituting the equation 2.3 and 2.4 into 2.2, we get the required result 2.1. □

Below theorem are the immediate consequences of the definitions 1.11, 1.12 and 1.19 and hence they are given without proof here.

Theorem 2.2. *Let* $\lambda \geqslant 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ *and* $m-1 \leqslant \Re(\mu) \leqslant m$, *then the following result holds:*

$$
{}^{\Gamma}I^{m,\,n+1}_{p+1,\,q+1}\left[az^{-\sigma}\,\middle|\,\begin{matrix}(\mathfrak{e}_1,\nu_1;\mathcal{E}_1:x),(\mathfrak{e}_j,\nu_j;\mathcal{E}_j)_{2,p},(\lambda-\mu,\sigma;1)\\(\lambda,\sigma;1),(\mathfrak{f}_j,\omega_j;\mathcal{F}_j)_{1,q}\end{matrix}\right]
= \sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\,{}^{\Gamma}I^{m,\,n+1}_{p+1,\,q+1}\left[az^{-\sigma}\,\middle|\,\begin{matrix}(\mathfrak{e}_1,\nu_1;\mathcal{E}_1:x),(\mathfrak{e}_j,\nu_j;\mathcal{E}_j)_{2,p},(1-k,\sigma;1)\\(1,\sigma;1),(\mathfrak{f}_j,\omega_j;\mathcal{F}_j)_{1,q}\end{matrix}\right]. \tag{2.5}
$$

Theorem 2.3. *Let* $\lambda \geqslant 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ *and* $m-1 \leqslant \Re(\mu) \leqslant m$, *then the following result holds:*

$$
{}^{\gamma}I^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{matrix}(\mathfrak{e}_1,\nu_1;\mathcal{E}_1:x),(1-\lambda,\sigma;1),(\mathfrak{e}_j,\nu_j;\mathcal{E}_j)_{2,p}\\(\mathfrak{f}_j,\omega_j;\mathcal{F}_j)_{1,q},(1-\lambda+\mu,\sigma;1)\end{matrix}\right]
= \sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\,{}^{\gamma}I^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{matrix}(\mathfrak{e}_1,\nu_1;\mathcal{E}_1:x),(0,\sigma;1),(\mathfrak{e}_j,\nu_j;\mathcal{E}_j)_{2,p}\\(\mathfrak{f}_j,\omega_j;\mathcal{F}_j)_{1,q},(k,\sigma;1)\end{matrix}\right]. \tag{2.6}
$$

Theorem 2.4. *Let* $\lambda \geqslant 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ *and* $m-1 \leqslant \Re(\mu) \leqslant m$, *then the following result holds:*

$$
{}^{\gamma}I^{m,\,n+1}_{p+1,\,q+1}\left[az^{-\sigma}\,\middle|\,\begin{matrix}(\mathfrak{e}_1,\nu_1;\mathcal{E}_1:x),(\mathfrak{e}_j,\nu_j;\mathcal{E}_j)_{2,p},(\lambda-\mu,\sigma;1)\\(\lambda,\sigma;1),(\mathfrak{f}_j,\omega_j;\mathcal{F}_j)_{1,q}\end{matrix}\right]
= \sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\,{}^{\gamma}I^{m,\,n+1}_{p+1,\,q+1}\left[az^{-\sigma}\,\middle|\,\begin{matrix}(\mathfrak{e}_1,\nu_1;\mathcal{E}_1:x),(\mathfrak{e}_j,\nu_j;\mathcal{E}_j)_{2,p},(1-k,\sigma;1),\\(1,\sigma;1),(\mathfrak{f}_j,\omega_j;\mathcal{F}_j)_{1,q}\end{matrix}\right]. \tag{2.7}
$$

3. Special Cases

In this section, we determine a few expansion formulae for the incomplete $\bar{I}$-function, incomplete $\overline{H}$-function and incomplete H-function, as special instances of Theorem 2.1 and Theorem 2.2. For the meanings of incomplete $\overline{H}$-function and incomplete H-function one could see Srivastava [2]. To delineate the uses of fundamental outcomes, if we give the specific qualities to the parameters, then we obtain the following special cases:

(i) Incomplete $\bar{I}$-function: If we set $\mathcal{F}_j = 1$ $(j = 1, \cdots, m)$ in 1.12 and utilizing the connection, that is

$$
{}^{\Gamma}\bar{I}^{m,\,n}_{p,\,q}(z) = {}^{\Gamma}I^{m,\,n}_{p,\,q}\left[z\,\middle|\,\begin{matrix}(\mathfrak{e}_1,\nu_1;\mathcal{E}_1,x),(\mathfrak{e}_j,\nu_j;\mathcal{E}_j)_{2,p}\\(\mathfrak{f}_j,\omega_j;1)_{1,m},(\mathfrak{f}_j,\omega_j,\mathcal{F}_j)_{m+1,q}\end{matrix}\right], \tag{3.1}
$$

in 2.1 and 2.5, respectively, at that point we get the accompanying corollaries:

Corollary 3.1. *Let* $\lambda \geqslant 1$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ *and* $m-1 \leqslant \Re(\mu) \leqslant m$, *then the following formula holds:*

$$
\begin{aligned}
&{}^{\Gamma}\bar{I}^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{matrix}(e_1,\nu_1;\mathcal{E}_1:x),(1-\lambda,\sigma;1),(e_j,\nu_j;\mathcal{E}_j)_{2,p}\\(f_j,\omega_j;1)_{1,m},(f_j,\omega_j;\mathcal{F}_j)_{m+1,q},(1-\lambda+\mu,\sigma;1)\end{matrix}\right]\\
&=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\,{}^{\Gamma}\bar{I}^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{matrix}(e_1,\nu_1;\mathcal{E}_1:x),(0,\sigma;1),(e_j,\nu_j;\mathcal{E}_j)_{2,p}\\(f_j,\omega_j;1)_{1,m},(f_j,\omega_j;\mathcal{F}_j)_{m+1,q},(k,\sigma;1)\end{matrix}\right].
\end{aligned}
\tag{3.2}
$$

Corollary 3.2. *Let* $\lambda \geqslant 1$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ *and* $m-1 \leqslant \Re(\mu) \leqslant m$, *then the following result holds:*

$$
\begin{aligned}
&{}^{\Gamma}\bar{I}^{m,\,n+1}_{p+1,\,q+1}\left[az^{-\sigma}\,\middle|\,\begin{matrix}(e_1,\nu_1;\mathcal{E}_1:x),(e_j,\nu_j;\mathcal{E}_j)_{2,p},(\lambda-\mu,\sigma;1)\\(\lambda,\sigma;1),(f_j,\omega_j;1)_{1,m},(f_j,\omega_j;\mathcal{F}_j)_{m+1,q}\end{matrix}\right]\\
&=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\,{}^{\Gamma}\bar{I}^{m,\,n+1}_{p+1,\,q+1}\left[az^{-\sigma}\,\middle|\,\begin{matrix}(e_1,\nu_1;\mathcal{E}_1:x),(e_j,\nu_j;\mathcal{E}_j)_{2,p},(1-k,\sigma;1)\\(1,\sigma;1),(f_j,\omega_j;1)_{1,m},(f_j,\omega_j;\mathcal{F}_j)_{m+1,q}\end{matrix}\right].
\end{aligned}
\tag{3.3}
$$

(ii) Incomplete $\overline{H}$-function $\overline{\Gamma}^{m,\,n}_{p,\,q}(z)$: if we give particular values to the parameters, such as $\mathcal{F}_j$ $(j=1,\cdots,m)=1$ and $\mathcal{E}_j$ $(j=n+1,\cdots,p)=1$ in 1.12, and using the below relation (see [2])

$$
\begin{aligned}
\overline{\Gamma}^{m,\,n}_{p,\,q}(z) &= {}^{\Gamma}I^{m,\,n}_{p,\,q}\left[z\,\middle|\,\begin{matrix}(e_1,\nu_1;\mathcal{E}_1:x),(e_j,\nu_j;\mathcal{E}_j)_{2,n},(e_j,\nu_j;1)_{n+1,p}\\(f_j,\omega_j;1)_{1,m},(f_j,\omega_j;\mathcal{F}_j)_{m+1,q}\end{matrix}\right]\\
&=\overline{\Gamma}^{m,\,n}_{p,\,q}\left[z\,\middle|\,\begin{matrix}(e_1,\nu_1;\mathcal{E}_1:x),(e_j,\nu_j;\mathcal{E}_j)_{2,n},(e_j,\nu_j)_{n+1,p}\\(f_j,\omega_j)_{1,m},(f_j,\omega_j;\mathcal{F}_j)_{m+1,q}\end{matrix}\right],
\end{aligned}
\tag{3.4}
$$

we get the following corollaries:

Corollary 3.3. *Let* $\lambda \geqslant 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ *and* $m-1 \leqslant \Re(\mu) \leqslant m$, *then the following result holds:*

$$
\begin{aligned}
&\overline{\Gamma}^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{matrix}(e_1,\nu_1;\mathcal{E}_1:x),(1-\lambda,\sigma;1),(e_j,\nu_j;\mathcal{E}_j)_{2,n},\\(f_j,\omega_j)_{1,m},(f_j,\omega_j;\mathcal{F}_j)_{m+1,q},(1-\lambda+\mu,\sigma;1)\end{matrix}\right]\\
&=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\times\\
&\qquad\overline{\Gamma}^{m,\,n+1}_{p+1,\,q+1}\left[az^{\sigma}\,\middle|\,\begin{matrix}(e_1,\nu_1;\mathcal{E}_1:x),(0,\sigma;1),(e_j,\nu_j;\mathcal{E}_j)_{2,n},(e_j,\nu_j)_{n+1,p}\\(f_j,\omega_j)_{1,m},(f_j,\omega_j;\mathcal{F}_j)_{m+1,q},(k,\sigma;1)\end{matrix}\right].
\end{aligned}
\tag{3.5}
$$

Corollary 3.4. *Let* $\lambda \geqslant 1$, $\sigma > 0$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ *and* $m-1 \leqslant \Re(\mu) \leqslant m$, *then the following result holds:*

$$\overline{\Gamma}^{m,\,n+1}_{p+1,\,q+1}\left[az^{-\sigma}\left|\begin{array}{c}(e_1,\nu_1;\mathcal{E}_1:x),(e_j,\nu_j;\mathcal{E}_j)_{2,n},(e_j,\nu_j)_{n+1,p},(\lambda-\mu,\sigma)\\(\lambda,\sigma),(f_j,\omega_j)_{1,m},(f_j,\omega_j;\mathcal{F}_j)_{m+1,q}\end{array}\right.\right]$$
$$=\sum_{k=0}^{\infty}\binom{\mu}{k}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu+k)}\times$$
$$\overline{\Gamma}^{m,\,n+1}_{p+1,\,q+1}\left[az^{-\sigma}\left|\begin{array}{c}(e_1,\nu_1;\mathcal{E}_1:x),(e_j,\nu_j;\mathcal{E}_j)_{2,n},(e_j,\nu_j)_{n+1,p},(1-k,\sigma)\\(1,\sigma),(f_j,\omega_j)_{1,m},(f_j,\omega_j;\mathcal{F}_j)_{m+1,q}\end{array}\right.\right]. \tag{3.6}$$

(iii) Incomplete H-function $\Gamma^{m,\,n}_{p,\,q}(z)$**:** if we put $\mathcal{F}_j\ (j=1,\cdots,q)=1$ and $\mathcal{E}_j\ (j=1,\cdots,p)=1$ in 1.12, and utilizing the connection, that is (see [2])

$$\Gamma^{m,\,n}_{p,\,q}(z) = {}^{\Gamma}I^{m,\,n}_{p,\,q}\left[z\left|\begin{array}{c}(e_1,\nu_1;1:x),(e_j,\nu_j;1)_{2,p}\\(f_j,\omega_j;1)_{1,q}\end{array}\right.\right]$$
$$=\Gamma^{m,\,n}_{p,\,q}\left[z\left|\begin{array}{c}(e_1,\nu_1:x),(e_j,\nu_j)_{2,p}\\(f_j,\omega_j)_{1,q}\end{array}\right.\right], \tag{3.7}$$

in 2.1 and 2.5 respectively, then we get the results obtained by Meena et al. [17, Theorem 2.1. and Theorem 2.2.].

Remark: Similarly, special cases for the Theorem 2.3 and Theorem 2.4 may be derived.

4. Conclusion

In this paper, by making use of the Leibniz rule for the Riemann-Liouville type derivatives, we have obtained certain expansion formulae of the incomplete I-functions. For our main findings, we have discussed some special cases and some known results also appeared.

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References

[1] Rathie AK (1997). *A new generalization of generalized Hypergeometric functions*. Le Math. **LII**: 297-310.
[2] Srivastava HM, Saxena RK and Parmar RK (2018). *Some families of the incomplete H-functions and the incomplete $\overline{H}$-functions and associated integral transforms and operators of fractional calculus with applications*. Rus. J. Math. Phys. **25**: 116-138. https://doi.org/10.1134/S1061920818010119
[3] Jangid K, Bhatter S, Meena S, Baleanu D and Purohit SD (2020). *Some fractional calculus findings associated with the incomplete I-functions*. Adv. Differ. Equ. **2020**: 265. https://doi.org/10.1186/s13662-020-02725-7

[4] Bansal MK, Kumar D, Singh J and Nisar KS (2020). *On the solutions of a class of integral equations pertaining to incomplete* H*-function and incomplete* $\overline{\mathrm{H}}$*-function*. Mathematics **8**(5): 819. https://doi.org/10.3390/math8050819

[5] Bansal MK, Kumar D, Nisar KS and Singh J (2020). *Certain fractional calculus and integral transform results of incomplete* $\aleph$*-functions with applications*. Math. Methods Appl. Sci. **43**(8): 5602-5614. https://doi.org/10.1002/mma.6299

[6] Bansal MK, Kumar D, Singh J, Tassaddiq A and Nisar KS (2020). *Some new results for the Srivastava-Luo-Raina* $\mathbb{M}$*-transform pertaining to the incomplete* H*-functions[J]*. AIMS Math. **5**(1): 717-722. https://doi.org/10.3934/math2020048

[7] Bansal MK, Kumar D, Khan I, Singh J and Nisar KS (2019). *Certain unified integrals associated with product of* M*-series and incomplete* H*-functions*. Mathematics **7**(12): 1191. https://doi.org/10.3390/math7121191

[8] Jangid K, Bhatter S, Meena S, Baleanu D, Al Qurashi M and Purohit SD (2020). *Some fractional calculus findings associated with the incomplete* I*-functions*. Adv. Differ Equ. 2020, 265. https://doi.org/10.1186/s13662-020-02725-7

[9] Meena S, Bhatter S, Jangid K and Purohit SD (2020). *Certain integral transforms concerning the product of family of polynomials and generalized incomplete functions*. Moroccan J. of Pure and Appl. Anal. (MJPAA) 6(2), 243–254. http://dx.doi.org/10.2478/mjpaa-2020-0019

[10] Baleanu D, Jangid NK, Joshi S and Purohit SD (2020). *The pathway fractional integrals of incomplete* I*-functions*. Int. J. Appl. Comput. Math,.6(5), 151, 8 pp. https://doi.org/10.1007/s40819-020-00902-6

[11] Jangid NK, Joshi S, Jangid K, Araci S and Purohit SD (2021). *Fractional calculus operators applied to the functions involving the product of Srivastava polynomials and incomplete I-functions*. Adv. Stud. Contemp. Math., Kyungshang. **31(2)**, 243-258. http://dx.doi.org/10.17777/ascm2021.31.2.243

[12] Rahman G, Saboor A, Anjum Z, Nisar KS and Abdeljawad T (2020). *An Extension of the Mittag-Leffler function and its associated properties*. Adv. Math. Phys. **8**. https://doi.org/10.1155/2020/5792853

[13] Ali RS, Mubeen S, Nayab I, Araci S, Rahman G and Nisar KS (2020). *Some fractional operators with the generalized Bessel-Maitland function*. Dis. Dyn. Nat. and Soc. **15**. https://doi.org/10.1155/2020/1378457

[14] Rahman G, Nisar KS and Arshad M (2020). *A new extension and applications of Caputo fractional derivative operator*. Analysis **1**. https://doi.org/10.1515/anly-2019-0005

[15] Kilbas AA, Srivastava HM and Trujillo JJ (2006). "Theory and Applications of Fractional Differential Equations". North-Holland Math Stud, Elsevier, Amsterdam, **204**.

[16] Osler TJ (1972). *A further extension of the Leibniz rule to fractional derivatives and its relation to Parseval's formula*. SIAM J. Math. Anal. **3**(1): 1-16. https://doi.org/10.1137/0503001

[17] Meena S, Bhatter S, Jangid K and Purohit SD (2020). *Certain expansion formulae of incomplete* H*-functions associated with leibniz rule*. TWMS. J. App. and Eng. Math. (accepted).