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Note on the weighted midpoint type inequalities having the Hölder condition

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Abstract

In this note, some new weighted midpoint type inequalities for Hölder continuous functions are given.

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2010 MSC: 26D10, 26D15, 26A51..

1. Introduction

Mathematical inequalities are a powerful and very important tool in many branches of mathematics such as the theory of differential and integral equations as well as the theory of approximations and numerical analysis. Due to their wide fields of application in various problems related to other sciences such as physics, biology and engineering in general. They have attracted the attention of many researchers who have given rise to several investigations and studies see for example [1, 2, 3, 4, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17], and references therein.

In [8], Kirmaci gave the following midpoint type inequalities

$$\left|\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\xi(\tau)\,d\tau-\xi\left(\frac{\sigma_1+\sigma_2}{2}\right)\right|\leqslant\frac{\sigma_2-\sigma_1}{8}\left(\left|\xi'(\sigma_1)\right|+\left|\xi'(\sigma_2)\right|\right),$$

and

$$\left|\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\xi(\tau)\,d\tau-\xi\left(\frac{\sigma_1+\sigma 2}{2}\right)\right|$$

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$$\leqslant \frac{\sigma_2-\sigma_1}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left(\left|\xi'(\sigma_1)\right|^{\frac{p}{p-1}}+3\left|\xi'(\sigma_2)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\left(3\left|\xi'(\sigma_1)\right|^{\frac{p}{p-1}}+\left|\xi'(\sigma_2)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.$$

In this note, we investigate some new weighted midpoint inequalities for functions having Hölder condition and for bounded functions.

2. Main results

We start by demonstrating this equality, then we will discuss our main results.

Lemma 2.1. *Let $\lambda:[\sigma_1,\sigma_2]\to\mathbb{R}$ be symmetric with respect to $\frac{\sigma_1+\sigma_2}{2}$, with $\sigma_1<\sigma_2$. And let $\xi:[\sigma_1,\sigma_2]\to\mathbb{R}$ be a differentiable function on (σ_1,σ_2). If $\xi,\lambda\in L([\sigma_1,\sigma_2])$, then*

$$\begin{aligned}&-\left(\int_{\sigma_1}^{\sigma_2}\lambda(z)\,dz\right)\xi\left(\tfrac{\sigma_1+\sigma_2}{2}\right)+\int_{\sigma_1}^{\sigma_2}\lambda(z)\,\xi(z)\,dz\\&=\frac{(\sigma_2-\sigma_1)^2}{4}\left(\int_0^1 p_1(\upsilon)\,\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)d\upsilon-\int_0^1 p_2(\upsilon)\,\xi'\left(\upsilon\sigma_1+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)d\upsilon\right),\end{aligned}$$

where

$$p_1(\upsilon)=\int_\upsilon^1\lambda\left(r\sigma_2+(1-r)\tfrac{\sigma_1+\sigma_2}{2}\right)dr \tag{2.1}$$

and

$$p_2(\upsilon)=\int_\upsilon^1\lambda\left(r\sigma_1+(1-r)\tfrac{\sigma_1+\sigma_2}{2}\right)dr. \tag{2.2}$$

Proof. Let

$$I=\int_0^1 p_1(\upsilon)\,\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)d\upsilon-\int_0^1 p_2(\upsilon)\,\xi'\left(\upsilon\sigma_1+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)d\upsilon. \tag{2.3}$$

Integrating by parts and changing the variables, we obtain

$$\begin{aligned}&\int_0^1 p_1(\upsilon)\,\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)d\upsilon\\&=\int_0^1\left(\int_\upsilon^1\lambda\left(r\sigma_2+(1-r)\tfrac{\sigma_1+\sigma_2}{2}\right)dr\right)\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)d\upsilon\\&=\frac{2}{\sigma_2-\sigma_1}\left(\int_\upsilon^1\lambda\left(r\sigma_2+(1-r)\tfrac{\sigma_1+\sigma_2}{2}\right)dr\right)\xi\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)\Bigg|_{\upsilon=0}^{\upsilon=1}\end{aligned}$$

$$+\frac{2}{\sigma_2-\sigma_1}\int_0^1 \lambda\left(\upsilon\sigma_2+(1-\upsilon)\frac{\sigma_1+\sigma_2}{2}\right)\xi\left(\upsilon\sigma_2+(1-\upsilon)\frac{\sigma_1+\sigma_2}{2}\right)d\upsilon$$

$$=-\frac{2}{\sigma_2-\sigma_1}\left(\int_0^1 \lambda\left(r\sigma_2+(1-r)\frac{\sigma_1+\sigma_2}{2}\right)dr\right)\xi\left(\frac{\sigma_1+\sigma_2}{2}\right)$$

$$+\frac{2}{\sigma_2-\sigma_1}\int_0^1 \lambda\left(\upsilon\sigma_2+(1-\upsilon)\frac{\sigma_1+\sigma_2}{2}\right)\xi\left(\upsilon\sigma_2+(1-\upsilon)\frac{\sigma_1+\sigma_2}{2}\right)d\upsilon$$

$$=-\left(\frac{2}{\sigma_2-\sigma_1}\right)^2\left(\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_2}\lambda(z)\,dz\right)\xi\left(\frac{\sigma_1+\sigma_2}{2}\right)+\left(\frac{2}{\sigma_2-\sigma_1}\right)^2\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_2}\lambda(z)\,\xi(z)\,dz. \tag{2.4}$$

Similarly, we have

$$\int_0^1 p_2(\upsilon)\,\xi'\left(\upsilon\sigma_1+(1-\upsilon)\frac{\sigma_1+\sigma_2}{2}\right)d\upsilon$$

$$=\left(\frac{2}{\sigma_2-\sigma_1}\right)^2\left(\int_{\sigma_1}^{\frac{\sigma_1+\sigma_2}{2}}\lambda(z)\,dz\right)\xi\left(\frac{\sigma_1+\sigma_2}{2}\right)-\left(\frac{2}{\sigma_2-\sigma_1}\right)^2\int_{\sigma_1}^{\frac{\sigma_1+\sigma_2}{2}}\lambda(z)\,\xi(z)\,dz. \tag{2.5}$$

Substituting (2.4) and (2.5) in (2.3), using the symmetry of λ, and then multiplying the result by $\frac{(\sigma_2-\sigma_1)^2}{4}$, we get the desired result. □

Theorem 2.2. *Let $\lambda:[\sigma_1,\sigma_2]\to\mathbb{R}$ be symmetric with respect to $\frac{\sigma_1+\sigma_2}{2}$, with $\sigma_1<\sigma_2$. And let $\xi:[\sigma_1,\sigma_2]\to\mathbb{R}$ be a differentiable function on (σ_1,σ_2) such that $\xi'\in L([\sigma_1,\sigma_2])$. If there exist constants $\varphi<\Phi$ such that $-\infty<\varphi\leqslant\xi'(u)\leqslant\Phi<+\infty$ for all $z\in[\sigma_1,\sigma_2]$, then we have*

$$|\Lambda(\sigma_1,\sigma_2,\lambda,\xi)|\leqslant\frac{(\Phi-\varphi)(\sigma_2-\sigma_1)^2}{8}\|\lambda\|_{[\sigma_1,\sigma_2],\infty},$$

where

$$\Lambda(\sigma_1,\sigma_2,\lambda,\xi)=\int_{\sigma_1}^{\sigma_2}\lambda(z)\,\xi(z)\,dz-\left(\int_{\sigma_1}^{\sigma_2}\lambda(z)\,dz\right)\xi\left(\frac{\sigma_1+\sigma_2}{2}\right)$$
$$-\frac{(\Phi+\varphi)(\sigma_2-\sigma_1)^2}{8}\left(\int_0^1 p_1(\upsilon)\,d\upsilon-\int_0^1 p_2(\upsilon)\,d\upsilon\right). \tag{2.6}$$

Proof. From Lemma 2.1, we have

$$\int_{\sigma_1}^{\sigma_2}\lambda(z)\,\xi(z)\,dz-\left(\int_{\sigma_1}^{\sigma_2}\lambda(z)\,dz\right)\xi\left(\frac{\sigma_1+\sigma_2}{2}\right)$$

$$
\begin{aligned}
&=\tfrac{(\sigma_2-\sigma_1)^2}{4}\left(\int_0^1 p_1(\upsilon)\,\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)d\upsilon\right.\\
&\quad\left.-\int_0^1 p_2(\upsilon)\,\xi'\left(\upsilon\sigma_1+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)d\upsilon\right)\\
&=\tfrac{(\sigma_2-\sigma_1)^2}{4}\left\{\int_0^1 p_1(\upsilon)\left(\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\tfrac{(\Phi-\varphi)}{2}+\tfrac{(\Phi-\varphi)}{2}\right)d\upsilon\right.\\
&\quad\left.-\int_0^1 p_2(\upsilon)\left(\xi'\left(\upsilon\sigma_1+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\tfrac{(\Phi-\varphi)}{2}+\tfrac{(\Phi-\varphi)}{2}\right)d\upsilon\right\}\\
&=\tfrac{(\sigma_2-\sigma_1)^2}{4}\left\{\int_0^1 p_1(\upsilon)\left(\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\tfrac{(\Phi-\varphi)}{2}\right)d\upsilon+\tfrac{(\Phi-\varphi)}{2}\int_0^1 p_1(\upsilon)\,d\upsilon\right.\\
&\quad\left.-\int_0^1 p_2(\upsilon)\left(\xi'\left(\upsilon\sigma_1+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\tfrac{(\Phi-\varphi)}{2}\right)d\upsilon-\tfrac{(\Phi-\varphi)}{2}\int_0^1 p_2(\upsilon)\,d\upsilon\right\}.
\end{aligned}
\tag{2.7}
$$

Thus, (2.7) gives

$$
\begin{aligned}
\Lambda(\sigma_1,\sigma_2,\lambda,\xi)=&\tfrac{(\sigma_2-\sigma_1)^2}{4}\left\{\int_0^1 p_1(\upsilon)\left(\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\tfrac{(\Phi-\varphi)}{2}\right)d\upsilon\right.\\
&\left.-\int_0^1 p_2(\upsilon)\left(\xi'\left(\upsilon\sigma_1+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\tfrac{(\Phi-\varphi)}{2}\right)d\upsilon\right\},
\end{aligned}
\tag{2.8}
$$

where $\Lambda(\sigma_1,\sigma_2,\lambda,\xi)$ is defined in (2.6). By applying the absolute value in both sides of (2.8), we get

$$
\begin{aligned}
|\Lambda(\sigma_1,\sigma_2,\lambda,\xi)|\leqslant&\tfrac{(\sigma_2-\sigma_1)^2}{4}\left\{\int_0^1 |p_1(\upsilon)|\left|\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\tfrac{(\Phi-\varphi)}{2}\right|d\upsilon\right.\\
&\left.+\int_0^1 |p_2(\upsilon)|\left|\xi'\left(\upsilon\sigma_1+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\tfrac{(\Phi-\varphi)}{2}\right|d\upsilon\right\}.
\end{aligned}
\tag{2.9}
$$

Since $\varphi\leqslant\xi'(z)\leqslant\Phi$ for all $z\in[\sigma_1,\sigma_2]$, we have

$$
-\tfrac{\Phi-\varphi}{2}\leqslant\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\tfrac{(\Phi-\varphi)}{2}\leqslant\tfrac{\Phi-\varphi}{2},
$$

which implies

$$
\left|\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\tfrac{(\Phi-\varphi)}{2}\right|\leqslant\tfrac{\Phi-\varphi}{2}
\tag{2.10}
$$

and

$$\left|\xi'\left(\upsilon\sigma_1+(1-\upsilon)\,\tfrac{\sigma_1+\sigma_2}{2}\right)-\tfrac{(\Phi-\varphi)}{2}\right|\leqslant\tfrac{\Phi-\varphi}{2}. \tag{2.11}$$

Using (2.1), (2.2), (2.10) and (2.11) in (2.9), and the symmetry of w, we get

$$\begin{aligned}
|\Lambda(\sigma_1,\sigma_2,\lambda,\xi)| &\leqslant \tfrac{(\Phi-\varphi)(\sigma_2-\sigma_1)^2}{8}\left(\int_0^1\left|\int_\upsilon^1\lambda\left(r\sigma_2+(1-r)\,\tfrac{\sigma_1+\sigma_2}{2}\right)dr\right|d\upsilon\right.\\
&\quad\left.+\int_0^1\left|\int_\upsilon^1\lambda\left(r\sigma_1+(1-r)\,\tfrac{\sigma_1+\sigma_2}{2}\right)dr\right|d\upsilon\right)\\
&\leqslant \tfrac{(\Phi-\varphi)(\sigma_2-\sigma_1)^2}{8}\,\|\lambda\|_{[\sigma_1,\sigma_2],\infty}\left(\int_0^1\left|\int_\upsilon^1 dr\right|d\upsilon+\int_0^1\left|\int_\upsilon^1 dr\right|d\upsilon\right)\\
&= \tfrac{(\Phi-\varphi)(\sigma_2-\sigma_1)^2}{4}\,\|\lambda\|_{[\sigma_1,\sigma_2],\infty}\left(\int_0^1(1-\upsilon)\,d\upsilon\right)\\
&= \tfrac{(\Phi-\varphi)(\sigma_2-\sigma_1)^2}{8}\,\|\lambda\|_{[\sigma_1,\sigma_2],\infty},
\end{aligned}$$

which is desired result. □

Corollary 2.3. *Taking* $\lambda(z)=\frac{1}{\sigma_2-\sigma_1}$, *Theorem 1 becomes*

$$\left|\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\lambda(z)\,\xi(z)\,dz-\xi\left(\tfrac{\sigma_1+\sigma_2}{2}\right)\right|\leqslant\tfrac{(\Phi-\varphi)(\sigma_2-\sigma_1)}{8}.$$

Our next result involve the Hölder continuous functions. We recall that a function $\xi:[\sigma_1,\sigma_2]\to\mathbb{R}$ is of r-H-Hölder, if

$$|\xi(\theta_1)-\xi(\theta_2)|\leqslant H\,|\theta_1-\theta_2|^r$$

holds for all $\theta_1,\theta_2\in(\sigma_1,\sigma_2)$, where $H>0$ and $r\in(0,1]$, (see [5]).

Theorem 2.4. *Let* $\lambda:[\sigma_1,\sigma_2]\to\mathbb{R}$ *be symmetric with respect to* $\frac{\sigma_1+\sigma_2}{2}$, *with* $\sigma_1<\sigma_2$. *And let* $\xi:[\sigma_1,\sigma_2]\to\mathbb{R}$ *be a differentiable function on* (σ_1,σ_2) *such that* $\xi'\in L([\sigma_1,\sigma_2])$. *If* ξ' *satisfies a Hölder condition for some* $H>0$ *and* $r\in(0,1]$, *then we have*

$$|\digamma(\sigma_1,\sigma_2,\lambda,\xi)|\leqslant\tfrac{2}{2+r}\left(\tfrac{\sigma_2-\sigma_1}{2}\right)^{2+r}H\,\|\lambda\|_{[\sigma_1,\sigma_2],\infty},$$

where

$$\begin{aligned}
\digamma(\sigma_1,\sigma_2,\lambda,\xi) &= \int_{\sigma_1}^{\sigma_2}\lambda(z)\,\xi(z)\,dz-\left(\int_{\sigma_1}^{\sigma_2}\lambda(z)\,dz\right)\xi\left(\tfrac{\sigma_1+\sigma_2}{2}\right)\\
&\quad-\tfrac{(\sigma_2-\sigma_1)^2}{4}\left(\xi'(\sigma_2)\int_0^1 p_1(\upsilon)\,d\upsilon-\xi'(\sigma_1)\int_0^1 p_2(\upsilon)\,d\upsilon\right).
\end{aligned} \tag{2.12}$$

Proof. Using Lemma 2.1, we deduce

$$\begin{aligned}
&\int_{\sigma_1}^{\sigma_2} \lambda(z)\,\xi(z)\,dz - \left(\int_{\sigma_1}^{\sigma_2} \lambda(z)\,dz\right)\xi\left(\tfrac{\sigma_1+\sigma_2}{2}\right) \\
&= \tfrac{(\sigma_2-\sigma_1)^2}{4}\left(\int_0^1 p_1(\upsilon)\,\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)d\upsilon\right. \\
&\quad \left. - \int_0^1 p_2(\upsilon)\,\xi'\left(\upsilon\sigma_1+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)d\upsilon\right) \\
&= \tfrac{(\sigma_2-\sigma_1)^2}{4}\left(\int_0^1 p_1(\upsilon)\left(\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\xi'(\sigma_2)+\xi'(\sigma_2)\right)d\upsilon\right. \\
&\quad \left. - \int_0^1 p_2(\upsilon)\left(\xi'\left(\upsilon\sigma_1+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\xi'(\sigma_1)+\xi'(\sigma_1)\right)d\upsilon\right\} \\
&= \tfrac{(\sigma_2-\sigma_1)^2}{4}\left\{\int_0^1 p_1(\upsilon)\left(\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\xi'(\sigma_2)\right)d\upsilon+\xi'(\sigma_2)\int_0^1 p_1(\upsilon)\,d\upsilon\right. \\
&\quad \left. - \int_0^1 p_2(\upsilon)\left(\xi'\left(\upsilon\sigma_1+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\xi'(\sigma_1)\right)d\upsilon-\xi'(\sigma_1)\int_0^1 p_2(\upsilon)\,d\upsilon\right\}. \qquad (2.13)
\end{aligned}$$

So, from (2.13) we get

$$\begin{aligned}
\digamma(\sigma_1,\sigma_2,\lambda,\xi) &= \tfrac{(\sigma_2-\sigma_1)^2}{4}\left\{\int_0^1 p_1(\upsilon)\left(\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\xi'(\sigma_2)\right)d\upsilon\right. \\
&\quad \left. - \int_0^1 p_2(\upsilon)\left(\xi'\left(\upsilon\sigma_1+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\xi'(\sigma_1)\right)d\upsilon\right\}, \qquad (2.14)
\end{aligned}$$

where $\digamma(\sigma_1,\sigma_2,\lambda,\xi)$ is defined in (2.12). By applying the absolute value in both sides of (2.14), we get

$$\begin{aligned}
|\digamma(\sigma_1,\sigma_2,\lambda,\xi)| &\leqslant \tfrac{(\sigma_2-\sigma_1)^2}{4}\left\{\int_0^1 |p_1(\upsilon)|\left|\xi'\left(\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\xi'(\sigma_2)\right|d\upsilon\right. \\
&\quad \left. + \int_0^1 |p_2(\upsilon)|\left|\xi'\left(\upsilon\sigma_1+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}\right)-\xi'(\sigma_1)\right|d\upsilon\right\}. \qquad (2.15)
\end{aligned}$$

Since $\xi'$is a Hölder continuous function, from (2.15), we get

$$
\begin{aligned}
|\mathcal{F}(\sigma_1,\sigma_2,\lambda,\xi)| &\leqslant \tfrac{(\sigma_2-\sigma_1)^2}{4} H \left(\int_0^1 |p_1(\upsilon)| \left|\upsilon\sigma_2+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}-\sigma_2\right|^r d\upsilon \right. \\
&\quad \left. + \int_0^1 |p_2(\upsilon)| \left|\upsilon\sigma_1+(1-\upsilon)\tfrac{\sigma_1+\sigma_2}{2}-\sigma_1\right| d\upsilon \right) \\
&= \left(\tfrac{\sigma_2-\sigma_1}{2}\right)^{2+r} H \left(\int_0^1 |p_1(\upsilon)| (1-\upsilon)^r d\upsilon + \int_0^1 |p_2(\upsilon)| (1-\upsilon)^r d\upsilon \right).
\end{aligned} \tag{2.16}
$$

Substituting (2.1) and (2.2) in (2.16), and using the symmetry of λ, we obtain

$$
\begin{aligned}
|\mathcal{F}(\sigma_1,\sigma_2,\lambda,\xi)| &\leqslant \left(\tfrac{\sigma_2-\sigma_1}{2}\right)^{2+r} H \left(\int_0^1 \left| \int_\upsilon^1 \lambda\left(r\sigma_2+(1-r)\tfrac{\sigma_1+\sigma_2}{2}\right) dr \right| (1-\upsilon)^r d\upsilon \right. \\
&\quad \left. + \int_0^1 \left| \int_\upsilon^1 \lambda\left(r\sigma_1+(1-r)\tfrac{\sigma_1+\sigma_2}{2}\right) dr \right| (1-\upsilon)^r d\upsilon \right) \\
&\leqslant \left(\tfrac{\sigma_2-\sigma_1}{2}\right)^{2+r} H \|\lambda\|_{[\sigma_1,\sigma_2],\infty} \left(\int_0^1 \left| \int_\upsilon^1 dr \right| (1-\upsilon)^r d\upsilon + \int_0^1 \left| \int_\upsilon^1 dr \right| (1-\upsilon)^r d\upsilon \right) \\
&= \left(\tfrac{\sigma_2-\sigma_1}{2}\right)^{2+r} H \|\lambda\|_{[\sigma_1,\sigma_2],\infty} \left(2\int_0^1 (1-\upsilon)^{1+r} d\upsilon \right) \\
&= \left(\tfrac{\sigma_2-\sigma_1}{2}\right)^{2+r} H \|\lambda\|_{[\sigma_1,\sigma_2],\infty} \left(\tfrac{2}{2+r}\right) \\
&= \tfrac{2}{2+r} \left(\tfrac{\sigma_2-\sigma_1}{2}\right)^{2+r} H \|\lambda\|_{[\sigma_1,\sigma_2],\infty},
\end{aligned}
$$

which is desired result. □

Corollary 2.5. *Under the assumptions of Theorem 2.4, and if ξ' satisfies the Lipschitz condition for some $L > 0$, we obtain*

$$
|\mathcal{F}(\sigma_1,\sigma_2,\lambda,\xi)| \leqslant \tfrac{(\sigma_2-\sigma_1)^3}{12} L \|\lambda\|_{[\sigma_1,\sigma_2],\infty},
$$

Corollary 2.6. *Takeing $\lambda(z) = \frac{1}{\sigma_2-\sigma_1}$, Theorem 2.4 becomes*

$$
\left| \tfrac{1}{\sigma_2-\sigma_1} \int_{\sigma_1}^{\sigma_2} \xi(z)\, dz - \xi\left(\tfrac{\sigma_1+\sigma_2}{2}\right) \right| \leqslant \tfrac{(\sigma_2-\sigma_1)^{1+r}}{(2+r)2^{1+r}} H + \tfrac{\sigma_2-\sigma_1}{8} \left(\xi'(\sigma_2)-\xi'(\sigma_1)\right).
$$

Corollary 2.7. *Takeing* $\lambda(z) = \frac{1}{\sigma_2 - \sigma_1}$, *Corollary 2.5 becomes*

$$\left| \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \xi(z)\, dz - \xi\left(\frac{\sigma_1 + \sigma_2}{2}\right) \right| \leqslant \frac{(\sigma_2 - \sigma_1)^2}{12} L + \frac{\sigma_2 - \sigma_1}{8} \left(\xi'(\sigma_2) - \xi'(\sigma_1)\right).$$

3. Applications involving the arithmetic and logarithmic means

We recall that for arbitrary real numbers z, k.
The Arithmetic mean: $A(z,k) = \frac{z+k}{2}$.
The p-Logarithmic mean: $L_p(z,k) = \left(\frac{k^{p+1} - z^{p+1}}{(p+1)(k-z)}\right)^{\frac{1}{p}}$, $z, k > 0, z \neq k$ and $p \in \mathbb{R} \setminus \{0, -1\}$.

Proposition 3.1. *Let* $z, k \in \mathbb{R}$ *with* $0 < z < k$, *then we have*

$$\left| L_3^3(z,k) - A^3(z,k) \right| \leqslant \frac{3(k+z)(k-z)^2}{8}.$$

Proof. The assertion follows from Corollary 2.3, applied to the function $\xi(b) = b^3$ which $\xi'(b) = 3b^2$ and $3z^2 \leqslant \xi'(b) \leqslant 3k^2$ on $[z, k]$. □

Proposition 3.2. *Let* $z, k \in \mathbb{R}$ *with* $0 < z < k \leqslant 1$, *then we have*

$$\left| L_{\frac{3}{2}}^{\frac{3}{2}}(z,k) - A^{\frac{3}{2}}(z,k) \right| \leqslant \frac{(k-z)^{\frac{3}{2}}}{5\sqrt{2}} + \frac{3(k-z)}{16} \left(\sqrt{k} - \sqrt{z}\right).$$

Proof. The assertion follows from Corollary 2.6, applied to the function $\xi(b) = b^{\frac{3}{2}}$ which $\xi'(b) = \frac{3}{2} b^{\frac{1}{2}}$ is $\frac{1}{2}$-Hölder continuous function. □

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