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Fractional diffusion equation described by the Atangana-Baleanu fractional derivative and its approximate solution

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Abstract

In this paper, we propose the approximate solution of the fractional diffusion equation described by a non-singular fractional derivative. We use the Atangana-Baleanu-Caputo fractional derivative in our studies. The integral balance methods as the heat balance integral method introduced by Goodman and the double integral method developed by Hristov have been used for getting the approximate solution. In this paper, the existence and uniqueness of the solution of the fractional diffusion equation have been provided. We analyze the impact of the fractional operator in the diffusion process. We represent graphically the approximate solution of the fractional diffusion equation.

Keywords: Fractional diffusion equations, Approximate solutions, Atangana-Baleanu, Fractional derivative operator.
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1. Introduction

Nowadays, fractional calculus has received many attractions. This attraction is due to the various fractional derivatives used in the fields of fractional calculus. They are the good compromises for describing the physical phenomena. There exist many types of fractional derivative operators which are equivalents. The fractional derivatives with a non-singular kernel as: the Atangana-Baleanu fractional derivative [1], the fractional operators with generalized Mittag-Leffler kernels, see in [2, 3, 4], the Caputo Fabrizio fractional derivative [5]. The fractional derivatives with a singular kernel as: the Riemann-Liouville fractional derivative [6], the Caputo fractional derivative [6], the generalized forms of the Caputo fractional derivative and the Riemann-Liouville fractional derivative

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operators in [6, 7] proposed by Thabet et al., the conformable fractional derivative introduced by Khallil in [8]. The discrete version of the fractional operators with Mittag-Leffler kernels were recently introduced in the literature. In [9, 10, 11], Thabet et al. propose the fractional derivative operators with Mittag-Leffler kernels and thier integration by parts, existence and uniqueness of the solutions for initial value problems. In [12, 13, 14], Thabet et al. propose the discrete forms of the Atangana-Baleanu-Riemann and the Atangana-Baleanu-Caputo fractional differences, their monotonicity properties and integration by parts. The physical applications of the fractional calculus interest many authors [15, 16, 17, 34, 35]. In this paper, we substitute the ordinary derivative used in the second Fick equation by the Atangana-Baleanu fractional derivative. It is a suitable problem. Can we accept the substitution? The formalistic form used in this section, is it physically acceptable? Hristov started the physical interpretations of the fractional diffusion equation described by the Atangana Baleanu fractional derivative in Caputo sense in [18]. The author has found inadequacy of the fractional diffusion equation represented by the Atangana Baleanu fractional derivative in Caputo sense. Later in [15], Sene has proposed the analytical solution of the fractional diffusion equation described by the Atangana Baleanu fractional derivative in Caputo sense. Due to the form of the obtained solution. The author hasn't found the physical interpretation of the model. In this paper, we come with a new approximate solution of the fractional diffusion equation described by the Atangana Baleanu fractional derivative in Caputo sense. We use the integral balance methods as the heat balance integral method (HBIM) and the double integral method (DIM) [19, 20, 21, 31, 33]. The main contribution of this study is to give a potential physical interpretation of the fractional diffusion equation described by the Atangana-Baleanu fractional derivative in Caputo sense. We finish by validating this model for the physical future uses. In this paper, we also propose the optimal value of the exponent of the proposed approximate solution. In general, the problem consisting of getting the exponent is not trivial. Myers and Mitchell introduced two popular methods in [20, 22]. We have the matching method and the Myers criterion [20, 22]. Many investigations exist related to the heat balance integral method and the double integral method [19, 20, 22, 23, 24, 25, 26]. Myers, Mitchell, and Hristov did many of them. They find many results related to these methods, see in [27, 28, 29, 30]. The main ideas of the heat balance integral method and the double integral method is the use of the finite penetration depth. The finite penetration depth is a physical concept.

The paper is organized as follows: we recall the Atangana Baleanu fractional derivative and its properties in Section 2. In Section 3, we give the constructive equation to obtain, with Fick first equation and second equation, the fractional diffusion equation described by the Atangana-Baleanu fractional derivative. In Section 4, we described the basic properties used in the integral balance methods. In Section 5, we prove the existence and the uniqueness of the fractional diffusion equation represented by the Atangana-Baleanu fractional derivative in Caputo sense using Banach fixed point theorem. In Section 6, we propose the approximate solution of the fractional diffusion equation described by the Atangana-Baleanu fractional derivative in Caputo sense. In Section 7, we described the matching method for getting the exponent of the approximate solution of the fractional diffusion equation expressed by the Atangana-Baleanu fractional derivative in Caputo sense. In Section 8, we described briefly the Myers method of getting the exponent of the approximate solution of the fractional diffusion equation represented by the Atangana-Baleanu fractional derivative in Caputo sense. We finish with the conclusions in Section 9.

2. Derivative operators with non-singular kernels

In this section, we recall the necessaries definitions which we will use in our studies. For the rest of this section, let's the function v defined by $v : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \longrightarrow \mathbb{R}$.

The Antagana-Baleanu fractional derivative [1] for a function v, of order α is defined by

$$D_{\alpha}^{ABR} v(x,t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t v(x(s),s)E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right) ds, \tag{2.1}$$

for all $t > 0$, $E_{\alpha}(.)$ denotes the Mittag-Leffler function [1] with one parameter, and $\Gamma(.)$ represents the Euler Gamma function.

The Atangana-Baleanu derivative in the Caputo sense [1] of a given function v, of order α is defined as

$$D_{\alpha}^{ABC} v(x,t) = \frac{B(\alpha)}{1-\alpha}\int_0^t v'(x(s),s)E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right) ds, \tag{2.2}$$

for all $t > 0$, where $\Gamma(.)$ is Euler Gamma function and $E_{\alpha}(.)$ denotes the Mittag-Leffler function [1] with one parameter.

The Riemann-Liouville fractional integral [6, 32] of a given function v, of order α is defined as

$$I^{\alpha} v(x,t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} v(x(s),s) ds, \tag{2.3}$$

for all $t > 0$, where $\Gamma(.)$ represents the Gamma function.

The Atangana-Baleanu fractional integral [1] of a function v, of order α is defined as

$$I_{\alpha}^{AB} v(x,t) = \frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)} I_{\alpha}^{RL} v(x,t), \tag{2.4}$$

for all $t > 0$.

The Laplace transform of the Atangana-Baleanu fractional derivative [1], in the Caputo and the Riemann-Liouville sense, are defined as follows

$$\mathcal{L}\left\{D_{\alpha}^{ABC} v\right\}(s) = \frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha}\mathcal{L}\{v\} - s^{\alpha-1}v(0)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}. \tag{2.5}$$

Here $\mathcal{L}$ represents the usual Laplace transform.

$$\mathcal{L}\left\{D_{\alpha}^{ABR} v\right\}(s) = \frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha}\mathcal{L}\{v\}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}. \tag{2.6}$$

3. Mathematical modeling of the fractional diffusion equation

In this section, we present the constructive equations. We give the model which we will use later in our studies. We use the Fick first and second equations. The Fick first criterion related to the diffusion processes is defined as follows

$$G = -\mu \frac{\partial v}{\partial x}. \tag{3.1}$$

We pick the coefficient of the diffusion material $\mu = 1$ into Fick first equation (3.1), it follows that

$$G = -\frac{\partial v}{\partial x}. \tag{3.2}$$

The Fick first criterion represents the flux of the system. It is equivalent to the local density. The Fick second criterion in the context of the Atangana-Baleanu fractional derivative is given by

$${}_tD_\alpha^{ABC} = -\frac{\partial G}{\partial x}. \tag{3.3}$$

Replacing equation (3.2) into equation (3.3), we obtain the fractional diffusion equation in the context of the Atangana-Baleanu fractional derivative. It is expressed as the following form

$$\begin{aligned} D_\alpha^{ABC} &= -\frac{\partial G}{\partial x} \\ &= -\frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}\right) \\ &= \frac{\partial^2 v}{\partial x^2}. \end{aligned} \tag{3.4}$$

Finally, the fractional model of the diffusion equation described by the Atnagana-Baleanu fractional derivative is given by

$${}_tD_\alpha^{ABC} = \frac{\partial^2 v}{\partial x^2}. \tag{3.5}$$

The resolution of the fractional diffusion equation described by the fractional order derivative is the subject of many investigations. The numerical schemes and methods are proposed. They can be applied without any inconveniences with the Dirichlet and the Neumann boundary conditions. The problem consisting of finding the analytical solution of the diffusion equation is an open problem. There exist in the literature many proposed methods. But the uses of these methods depend on the types of the boundaries conditions. The Fourier sine transform and the Laplace transform can be used to solve the fractional diffusion equations, see in [15, 16, 17]. The uses of the Fourier sine transform and the Laplace transform [15, 16, 17] are adequate. They are simple to do, when you use the Dirichlet boundary conditions defined by $v(x,0) = 0$ and $v(0,t) = V_0$ (where V_0 designs a constant temperature). With the Neumann boundary conditions and the Dirichlet boundary condition expressed as $v(x,0) = f(x)$ and $v(0,t) = g(t)$, the use of the Fourier sine transform is not trivial and impossible in many cases. It is the limitation of this method. There exist the Homotopy method introduced by Liao in [**?**]. The technique is useful for getting the analytical approximation of the solution of the fractional diffusion equation. The application of the homotopy method take into account the boundary condition defined by $v(x,0) = f(x)$. It is simple to be applied when the function f is an exponential function or a polynomial function. In other cases, the application is not trivial and impossible in many cases. In this paper, we use the integral balance methods. What are the advantages and inconveniences of these methods? We will answer to these questions in the next sections.

4. Basics calculus for the integral balance methods

In this section, we give the basics calculations which we will use for applying the heat balance integral method and the double integral method. The integral balance methods provide an approximate analytical solution (semi-analytical solution) of the diffusion equations. For that, the form of the approximate solution of the fractional diffusion equation is expressed in the following form

$$v(x,t) = \left(1 - \frac{x}{\delta}\right)^n, \tag{4.1}$$

where δ represents the finite penetration depth. The above approximate solution is valid for the parabolic equation which satisfies the called Goodman conditions. Let's recall the following calculations. The first relation is represented in the following relationships

$$\begin{aligned} \int_0^\delta D_\alpha^{ABC} v(x,t)dx &= D_\alpha^{ABC} \int_0^\delta \left(1 - \frac{x}{\delta}\right)^n dx \\ &= D_\alpha^{ABC} \left[-\frac{\delta}{n+1}\left(1 - \frac{x}{\delta}\right)^{n+1} \right]_0^\delta \\ &= D_\alpha^{ABC} \left[\frac{\delta}{n+1} \right] \\ &= \frac{1}{n+1} D_\alpha^{ABC} \delta. \end{aligned} \tag{4.2}$$

The second relation is represented in the following relationships

$$\begin{aligned} \int_0^\delta \frac{\partial^2 v}{\partial x^2} dx &= -\frac{\partial v}{\partial x}_{\|x=0} \\ &= \frac{n}{\delta}\left(1 - \frac{x}{\delta}\right)^{n-1}_{\|x=0} \\ &= \frac{n}{\delta}. \end{aligned} \tag{4.3}$$

The third relation is represented in the following relationships

$$\begin{aligned}\int_0^\delta\int_x^\delta D_\alpha^{ABC}v(x,t)dxdx &= D_\alpha^{ABC}\int_0^\delta\int_x^\delta\left(1-\frac{x}{\delta}\right)^n dxdx\\ &= D_\alpha^{ABC}\int_0^\delta\frac{\delta}{n+1}\left(1-\frac{x}{\delta}\right)^{n+1}dx\\ &= D_\alpha^{ABC}\left[-\frac{\delta^2}{(n+1)(n+2)}\left(1-\frac{x}{\delta}\right)^{n+1}\right]_0^\delta\\ &= \frac{1}{(n+1)(n+2)}D_\alpha^{ABC}\delta^2. \end{aligned}\tag{4.4}$$

The fourth relation is represented in the following relationships

$$\begin{aligned}\int_0^\delta\int_x^\delta\frac{\partial^2 v}{\partial x^2}dxdx &= \int_0^\delta\left[\frac{\partial v}{\partial x}\right]_x^\delta dx\\ &= -\int_0^\delta\frac{\partial v}{\partial x}dx\\ &= v(0,t).\end{aligned}\tag{4.5}$$

We use equations (4.2) and (4.3) in the application of the heat balance integral method. We use equations (4.4) and (4.5) for the double integral method.

5. Existence and uniqueness of the fractional diffusion equation

In this section, we prove using Banach fixed theorem the existence and the uniqueness of the solution of the fractional diffusion equation. Let's the function

$$\psi(x,t,v)=\frac{\partial^2 v(x,t)}{\partial x^2}.\tag{5.1}$$

Firstly, let's prove the function ψ is Lipschitz continuous with a Lipschitz constant k_1. We assume the function v is bounded. Applying the norm there exists k_1 such that we have the following relationships

$$\begin{aligned}\|\psi(x,t,u)-\psi(x,t,v)\| &= \left\|\frac{\partial^2 u(x,t)}{\partial x^2}-\frac{\partial^2 v(x,t)}{\partial x^2}\right\|\\ &\leqslant k_1\|u(x,t)-v(x,t)\|.\end{aligned}\tag{5.2}$$

Thus, under the assumption the function v is bounded, the function ψ is Lipschitz continuous with a Lipschitz constant k_1. The second step, we apply the Atangana-Baleanu fractional integral on the fractional diffusion equation (5.1), we obtain the following equation

$$v(x,t)-v(x,0)=\frac{1-\alpha}{B(\alpha)}\psi(x,t,v)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\psi(x,t,v)\,ds.\tag{5.3}$$

Let's the map defined by $Tv: H\to H$, where H is a closed set subspace of a Banach space and the function Tv is defined by

$$Tv(x,t)=\frac{1-\alpha}{B(\alpha)}\psi(x,t,v)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\psi(x,t,v)\,ds.\tag{5.4}$$

Firstly, we prove the operator posed in Eq. (5.4) is well definite. We apply again the euclidean norm to the following equation

$$\begin{aligned}\|Tv(x,t)-v(x,0)\| &= \left\|\frac{1-\alpha}{B(\alpha)}\psi\left(x,t,v\right)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\psi\left(x,t,v\right)ds\right\| \\ &\leqslant \frac{1-\alpha}{B(\alpha)}\left\|\psi\left(x,t,v\right)\right\|+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|\psi\left(x,t,v\right)\right\|ds \\ &\leqslant \frac{1-\alpha}{B(\alpha)}\left\|\psi\left(x,t,v\right)\right\|+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\left\|\psi\left(x,t,v\right)\right\|\int_0^t (t-s)^{\alpha-1}ds \\ &\leqslant \frac{1-\alpha}{B(\alpha)}M+\frac{a^{\alpha}}{B(\alpha)\Gamma(\alpha)}M, \end{aligned} \tag{5.5}$$

where $t \leqslant a$ and the Lipchitz constant M comes from the fact ψ is Lipchitz contonous. Thus, the operator T is well defined.

We provided a condition under which the operator T is a contraction. We have the following majoration and the fact the function v is bounded

$$\|Tu(x,t)-Tv(x,t)\| \leqslant \left[\frac{1-\alpha}{B(\alpha)}k_1+\frac{\alpha k_1\delta^{\alpha}}{B(\alpha)\Gamma(\alpha)}\right]\|u(x,t)-v(x,t)\|. \tag{5.6}$$

Thus, under the condition

$$\frac{1-\alpha}{B(\alpha)}k_1+\frac{\alpha k_1\delta^{\alpha}}{B(\alpha)\Gamma(\alpha)}<1. \tag{5.7}$$

Then the operator T is a contraction. Using the Banach Fixed Point Theorem, the fractional diffusion equation described by the Atangana-Baleanu fractional derivative has a unique solution. We know the exact solution exists and is unique. Thus in the next section, we will propose an approximate solution of the exact solution of the fractional diffusion equation. The existence and uniqueness of the solution justify the problem consisting of introducing an approximate solution.

6. Approximation with integral balance methods

In this section, we present the approximate solution of the fractional diffusion equation described by the Atangana-Baleanu fractional derivative in Caputo sense. Let's the fractional diffusion equation expressed by the Atangana-Baleanu fractional derivative in Caputo sense represented as the following form

$$_tD_{\alpha}^{ABC}v(x,t)=\frac{\partial^2 v(x,t)}{\partial x^2}, \tag{6.1}$$

with the initial Dirichlet boundary conditions described by

- $v(x,0)=0$ for $x>0$,
- $v(0,t)=1$ for $t>0$.

We use the heat balance integral methods and the double integral method of getting the finite penetration depth.

6.1. Approximation with the heat balance integral method

In this section, we described the method of getting the finite penetration depth. The technique consists of integrating the fractional diffusion equation (6.1) defined by the Atangana-Baleanu-Caputo fractional derivative between 0 to the penetration depth δ.

$$\begin{aligned}\int_0^\delta D_\alpha^{ABC} v(x,t)dx &= \int_0^\delta \frac{\partial^2 v(x,t)}{\partial x^2}dx \\ \frac{1}{n+1}D_\alpha^{ABC}\delta &= \frac{n}{\delta} \\ \frac{1}{n+1}D_\alpha^{ABC} &= \frac{n}{\delta} \\ D_\alpha^{ABC}\delta &= \frac{n(n+1)}{\delta}. \end{aligned} \tag{6.2}$$

From equation (6.2), we multiply by the finite penetration depth δ. We obtain the following fractional differential equation

$$D_\alpha^{ABC}\delta^2 = 2n(n+1). \tag{6.3}$$

We use the Laplace transform of the Atangana-Balueanu-Caputo fractional derivative to both sides of equation (6.3). We have under assumption $\delta(0) = 0$, the following expressions

$$\begin{aligned}\frac{B(\alpha)}{1-\alpha}\frac{s^\alpha\bar{\delta}(s) - s^{\alpha-1}\delta(0)}{s^\alpha + \frac{\alpha}{1-\alpha}} &= \frac{2n(n+1)}{s} \\ \frac{B(\alpha)}{1-\alpha}\frac{s^\alpha\bar{\delta}(s)}{s^\alpha + \frac{\alpha}{1-\alpha}} &= \frac{2n(n+1)}{s} \\ \bar{\delta}^2(s) &= \frac{2n(n+1)(1-\alpha)}{B(\alpha)s} + \frac{2n(n+1)\alpha}{B(\alpha)s^{\alpha+1}}, \end{aligned} \tag{6.4}$$

where $\bar{\delta}$ represents the usual Laplace transformation of the function δ. We Apply the inverse of the Atangana-Baleanu fractional derivative in Caputo sense to both sides of equation (6.4). We obtain the following relationships

$$\begin{aligned}\delta^2(t) &= \frac{2n(n+1)(1-\alpha)}{B(\alpha)} + \frac{2n(n+1)\alpha t^\alpha}{B(\alpha)\Gamma(1+\alpha)} \\ \delta(t) &= \frac{\sqrt{2n(n+1)(1-\alpha)}}{\sqrt{B(\alpha)}} + \sqrt{2n(n+1)\alpha}\frac{t^{\frac{\alpha}{2}}}{\sqrt{B(\alpha)}\sqrt{\Gamma(\alpha+1)}}. \end{aligned} \tag{6.5}$$

The finite penetration depth of the fractional diffusion equation described by the Atangana-Balueanu fractional derivative in Caputo sense obtained with the HBIM method is given by

$$\delta(t) = \frac{\sqrt{2n(n+1)(1-\alpha)}}{\sqrt{B(\alpha)}} + \sqrt{2n(n+1)\alpha}\frac{t^{\frac{\alpha}{2}}}{\sqrt{B(\alpha)}\sqrt{\Gamma(\alpha+1)}}. \tag{6.6}$$

Using Atangana-Baleanu fractional integral the finite penetration depth of the fractional diffusion equation described by the Atangana-Balueanu fractional derivative in Caputo sense obtained with the heat balance integral method is given by

$$\delta(t) = \sqrt{2n(n+1)}\sqrt{I_\alpha^{AB}(1)(t)}. \tag{6.7}$$

The approximate solution of the fractional diffusion equation (6.1) described by the Atangana-Balueanu fractional derivative in Caputo sense is obtained by replacing the penetration depth δ into equation (4.1)

$$v(x,t) = \left(1-\frac{x}{\delta}\right)^n = \left(1-\frac{y}{\sqrt{2n(n+1)}\sqrt{I_\alpha^{AB}(1)(t)}}\right)^n. \tag{6.8}$$

The similarity variable is given by $x/\sqrt{I_\alpha^{AB}(1)(t)}$. In other words, we observe the Atangana-Baleanu fractional derivative generates it fractional integral in the similarity variable. We notice, using equation (6.8) and the fact $I_{\alpha=1}^{AB}(1)(t) = t$, we recover the penetration depth of the classical diffusion equation when $\alpha = 1$. That is, see in [19, 20, 22, 23, 24, 25, 26]

$$\delta(t) = \sqrt{2n(n+1)}\sqrt{t}. \tag{6.9}$$

The approximate solution of the classical diffusion equation described by the integer order derivative is given by

$$v(x,t) = \left(1-\frac{x}{\delta}\right)^n = \left(1-\frac{x}{\sqrt{2n(n+1)}\sqrt{t}}\right)^n. \tag{6.10}$$

The similarity variable is given by $x/\sqrt{t}$.

Let's discuss the validity of the fractional diffusion equation modeled with the Atangana-Baleanu fractional derivative in Caputo sense. We first notice when we use the Caputo fractional derivative the finite penetration depth is given by

$$\delta(t) = \sqrt{2n(n+1)}\sqrt{I_\alpha^{RL}(1)(t)}. \tag{6.11}$$

The similarity variable is given by $x/\sqrt{I_\alpha^{RL}(1)(t)}$. In other words, the Caputo fractional derivative generates its fractional integral into the similarity variable. When $\alpha = 1$, we recover the penetration depth of the classical diffusion equation. In other words, when we use the integer order derivative the finite penetration depth is given by

$$\delta(t) = \sqrt{2n(n+1)}\sqrt{I^{classical}(1)(t)} = \sqrt{2n(n+1)}\sqrt{t}. \tag{6.12}$$

The similarity variable is given by $x/\sqrt{I_1^{classical}(1)(t)}$. In other words, the integer order derivative generates its classical integral into the similarity variable.

In [18], the author tries to find a physical interpretation of the fractional diffusion equation described by the Atangana-Baleanu fractional derivative in Caputo sense. Due to the form of the penetration depth, the author hasn't found a physical interpretation of the fractional diffusion equation. The issue of this work contributes to give a possible interpretation of this model. The penetration depth of the fractional diffusion equation obtain with HBIM method is realistic. Its form is due to the form of the fractional integral of the Atangana-Baleanu fractional derivative. The function $x/\sqrt{I_\alpha^{AB}(1)(t)}$ represents the non Boltzmann similarity variable of the fractional diffusion equation. Thus, the Atangana-Baleanu fractional derivative in Caputo sense can be used for the modeling of the fractional diffusion equations.

In conclusion, the fractional derivative operator or the integer order derivative generates their integrals into the expression of the similarity variable. The Atangana-Baleanu fractional derivative represents a good compromise in physical modeling.

6.2. Approximation with the double integral method

In this section, we described the double integral method. The method consists of integrating the fractional diffusion equation represented by the Atangana-Baleanu fractional derivative in Caputo sense between y to the penetration depth δ. In second the integration, we integrate the obtained fractional differential equation in the first step between 0 to the penetration depth δ.

$$\begin{aligned} \int_0^\delta \int_x^\delta D_\alpha^{ABC} v(x,t)dx &= \int_0^\delta \int_x^\delta \frac{\partial^2 v(x,t)}{\partial x^2} dx \\ \frac{1}{(n+1)(n+2)} D_\alpha^{ABC} \delta^2 &= 1 \\ D_\alpha^{ABC} \delta^2 &= (n+1)(n+2) \\ D_\alpha^{ABC} \delta^2 &= (n+1)(n+2), \end{aligned} \tag{6.13}$$

We obtain the fractional differential equation described by the Atangana-Baleanu fractional derivative in Caputo sense defined.

$$D_\alpha^{ABC} \delta^2 = (n+1)(n+2). \tag{6.14}$$

We use the Laplace transform of the Atangana-Balueanu fractional derivative to both sides of equation (6.14). We have under assumption $\delta(0) = 0$, the following expressions

$$\begin{aligned} \frac{B(\alpha)}{1-\alpha} \frac{s^\alpha \bar{\delta}(s) - s^{\alpha-1}\delta(0)}{s^\alpha + \frac{\alpha}{1-\alpha}} &= \frac{(n+1)(n+2)}{s} \\ \frac{B(\alpha)}{1-\alpha} \frac{s^\alpha \bar{\delta}(s)}{s^\alpha + \frac{\alpha}{1-\alpha}} &= \frac{(n+1)(n+2)}{s} \\ \bar{\delta}^2(s) &= \frac{(n+1)(n+2)(1-\alpha)}{B(\alpha)s} + \frac{(n+1)(n+2)\alpha}{B(\alpha)s^{\alpha+1}}, \end{aligned} \tag{6.15}$$

where $\bar{\delta}$ represents the usual Laplace transformation of the function δ. We Apply the inverse of the Atangana-Baleanu fractional derivative Laplace transformation to both sides of equation (6.15). We obtain the following relationships

$$\begin{aligned} \delta^2(t) &= \frac{(n+1)(n+2)(1-\alpha)}{B(\alpha)} + \frac{(n+1)(n+2)\alpha t^\alpha}{B(\alpha)\Gamma(1+\alpha)} \\ \delta(t) &= \frac{\sqrt{(n+1)(n+2)(1-\alpha)}}{\sqrt{B(\alpha)}} + \sqrt{(n+1)(n+2)\alpha} \frac{t^{\frac{\alpha}{2}}}{\sqrt{B(\alpha)}\sqrt{\Gamma(\alpha+1)}}. \end{aligned} \tag{6.16}$$

The finite penetration depth of the fractional diffusion equation described by the Atangana-Balueanu fractional derivative in Caputo sense obtained with the double integral method is given by

$$\delta(t) = \frac{\sqrt{(n+1)(n+2)(1-\alpha)}}{\sqrt{B(\alpha)}} + \sqrt{(n+1)(n+2)\alpha} \frac{t^{\frac{\alpha}{2}}}{\sqrt{B(\alpha)}\sqrt{\Gamma(\alpha+1)}}. \tag{6.17}$$

We use the Atangana-Baleanu fractional integral. Thus, the finite penetration depth of the fractional diffusion equation described by the Atangana-Baleanu fractional derivative in Caputo sense obtained with the double integral method is given by

$$\delta(t) = \sqrt{(n+1)(n+2)}\sqrt{I_{\alpha}^{AB}\left(1\right)\left(t\right)}. \tag{6.18}$$

We replace the above penetration depth into equation (6.1). The approximate solution of the fractional diffusion equation described by the Atangana-Baleanu fractional derivative in Caputo sense is given

$$v(x,t) = \left(1-\frac{x}{\delta}\right)^{n} = \left(1-\frac{x}{\sqrt{(n+1)(n+2)}\sqrt{I_{\alpha}^{AB}\left(1\right)\left(t\right)}}\right)^{n}. \tag{6.19}$$

We use equation (6.17) and the fact $I_{\alpha=1}^{AB}\left(1\right)\left(t\right) = t$. We recover the penetration depth of the classical diffusion equation when $\alpha = 1$, see in [19, 20, 22, 23, 24, 25, 26]

$$\delta(t) = \sqrt{(n+1)(n+2)}\sqrt{t}. \tag{6.20}$$

The approximate solution of the classical diffusion equation described by the integer order derivative is given by

$$v(x,t) = \left(1-\frac{x}{\delta}\right)^{n} = \left(1-\frac{x}{\sqrt{(n+1)(n+2)}\sqrt{t}}\right)^{n}. \tag{6.21}$$

7. Matching method exponent n

In this section, we investigate the values of the exponent n of the approximate solution for the fractional diffusion equation described by the Atangana-Baleanu fractional derivative in Caputo sense, proposed in this paper. There exist many discussions related to the value of the exponent n, see in [20, 22]. There exist the Myers criterion method consisting of minimizing the Langford function [20, 22]. Another method proposed by Myers is called the matching criterion [22]. In the matching method, we consider the finite penetration depth generated by the HBIM and the finite penetration depth generated by the DIM are equals. In many investigations [20, 22], the authors choose $n = 2$ or $n = 3$. In this section, we use the matching method. Note the matching method stated by Myers in [22] was used and explained physically by Hristov in many works [19, 23, 25, 26]. We use this method in this section; we have the following relationships

$$\begin{aligned} \sqrt{2n(n+1)}\sqrt{I_{\alpha}^{RL}\left(1\right)\left(t\right)} &= \sqrt{(n+1)(n+2)}\sqrt{I_{\alpha}^{AB}\left(1\right)\left(t\right)} \\ \sqrt{2n(n+1)} &= \sqrt{(n+1)(n+2)} \\ 2n &= n+2 \\ n &= 2. \end{aligned} \tag{7.1}$$

In this case, we have the following approximate solution for the fractional diffusion equation (6.1) described by the Atangana-Baleanu fractional derivative. The finite penetration depth is obtained with HBIM method or the DIM method

$$v(x,t) = \left(1 - \frac{x}{\delta}\right)^2 = \left(1 - \frac{x}{2\sqrt{3}\sqrt{I_\alpha^{AB}(1)(t)}}\right)^2. \tag{7.2}$$

We depict in Figure 7, the approximate solution of the fractional diffusion equation described by the Atangana-Baleanu fractional derivative in three-dimensional space.

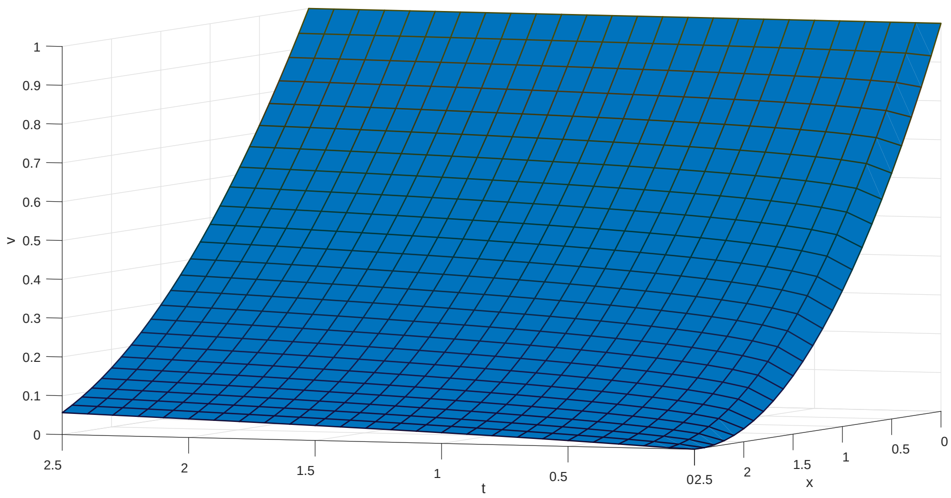

Figure 1: Approximate solutions of diffusion equation, $\alpha = 0.5$.

We consider the penetration depth obtained with the HBIM. We depict in Figure 7, the approximate solution of the fractional diffusion equation described by the Atangana-Baleanu fractional derivative in two-dimensional space, with $t = 0.6$ and for different values of the order α. We observe all the curves decrease and follow the increase of the order α. Thus we note a retardation effect.

8. Myers criterion related to the exponent n

In this section, we describe the Myers optimization principle of getting the exponent n briefly. We use the Langford function defined by

$$L_f = \int_0^1 \left[D_\alpha^{ABC} v(x,t) - \frac{\partial^2 v(x,t)}{\partial x^2} \right]^2 dx. \tag{8.1}$$

Myers in [22] proposes to minimize the above function of getting the exponent n. The problem consists of finding an optimal exponent n under which the Langford function is minimized. The criterion seems to be simple, but the applicability in the real problem is not trivial. But we are sure that the exponent found with Myers criterion is the best exponent n for the assumed profile of the fractional diffusion equation described by the Atangana-Baleanu fractional derivative in Caputo sense. Myers criterion is used in many

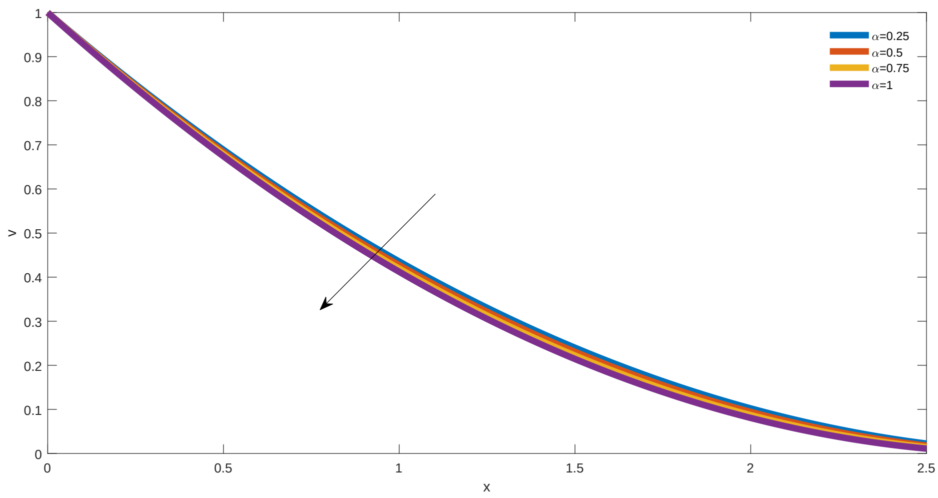

Figure 2: Approximate solutions of diffusion equation, different α and $t = 0.6$.

Hristov works related to the application of the integral balance methods, see in [19, 23, 25, 26]. Let's establish the explicit form of the Langford function. Let's the preliminary calculations. Firstly, the Atangana-Baleanu fractional derivative in the Caputo sense of the proposed approximate solution is given by

$$\begin{aligned} D_\alpha^{ABC} v(x,t) &= D_\alpha^{ABC} \left(1-\frac{x}{\delta}\right)^n \\ &= D_\alpha^{ABC} \left[1-\frac{nx}{\delta}+\frac{n(n+1)x^2}{2\delta^2}\right] \\ &= -nxD_\alpha^{ABC}\delta^{-1}+\frac{n(n+1)x^2}{2}D_\alpha^{ABC}\delta^{-2}. \end{aligned} \tag{8.2}$$

The following expression gives the second order derivative of the approximate solution.

$$\frac{\partial^2 v(x,t)}{\partial x^2} = \frac{n(n-1)}{\delta^2}\left(1-\frac{x}{\delta}\right)^{n-2}. \tag{8.3}$$

Using equation (8.2) and equation(8.3) into the Langford function, we get the function defined by

$$L_f = \int_0^1 \left[-nxD_\alpha^{ABC}\delta^{-1}+\frac{n(n+1)x^2}{2}D_\alpha^{ABC}\delta^{-2}-\frac{n(n-1)}{\delta^2}\left(1-\frac{x}{\delta}\right)^{n-2}\right]^2 dx, \tag{8.4}$$

where the finite penetration depth δ is obtained with the HBIM method or the DIM method. The main consequence of the Myers criterion are at the boundary condition; we have the particular values of the exponent n. That is the Goodman boundary condition must be satisfied. The Goodman boundary conditions are defined by

$$v(\delta,t) = \frac{\partial v(\delta,t)}{\partial x} = 0. \tag{8.5}$$

We notice when the function v is the exact solution of the fractional diffusion equation described by the Atangana-Baleanu fractional derivative in Caputo sense; then the Langford function is null, that is $L_f = 0$.

At the point $x = 0$, the the Langford function with the finite penetration depth obtained with the HBIM method or the DIM method, we have

$$L_f = \int_0^1 \left[\frac{n(n-1)}{\delta^2}\right]^2 = 0. \tag{8.6}$$

From which we get easily the exponent at the first boundary condition $n = 1$.

At the $x = \delta$. In other words, when the coordinate x converges to the penetration depth δ. In the way to satisfies the second Goodman boundary condition, we must impose the Langford function, when x approaches δ, must be positive or null. That is

$$\lim_{x\to\delta} \int_0^1 \left[\frac{n(n-1)}{\delta^2}\left(1-\frac{x}{\delta}\right)^{n-2}\right]^2 >= 0. \tag{8.7}$$

Then we obtain the exponent $n >= 2$.

The important remark is the exponents $n = 1$, and $n = 2$ got at the boundary conditions are not, in general, the optimal exponent n which minimize the Langford function. Alternatively, the method of finding the optimal exponent n when $x > 0$ and $x \neq \delta$ consists of applying the Myers criterion proposed above or see in Hristov's works [19, 23, 25, 26]. The problem is not trivial and is not the subject of this study. In this paper, we accept the value of the exponent $n = 2$ found in the matching method in the previous section.

9. Conclusion

In this paper, we have discussed the approximate solution of the fractional diffusion equation described by the Atangana-Baleanu fractional derivative in Caputo sense. We have used the heat balance integral method and the double integral method of getting the finite penetration depth. The main question is to bring the physical interpretation of the fractional diffusion equation described by the Atangana-Baleanu fractional derivative in Caputo sense for the physical use. We conclude the fractional diffusion equation represented by the Atangana-Baleanu fractional derivative in Caputo sense has a physical signification. The form of the similarity is due to the form of the fractional integral generated by the Atangana-Baleanu fractional derivative. For future works, we will focus the advantages of the generalized Atangana-Baleanu-Riemenn and Atangana-Baleanu-Caputo derivatives developped by Thabet et al. in context of the fractional diffusion equations.

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