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## Hyers-Ulam Stability Results for Tempered $(k, \psi)$ -Hilfer Fractional Differential Equations via the $(k, \psi)$ -Generalized Laplace Transform

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• Received: 16 January 2026

• Accepted: 09 June 2026

• Published Online: 30 June 2026

### Abstract

We study the Hyers-Ulam stability of tempered  $(k, \psi)$ -Hilfer fractional differential equations using the  $(k, \psi)$ -generalized Laplace transform. This transform plays a central role in deriving and extending stability results, underscoring its effectiveness in the analysis of tempered fractional operators. The theoretical contributions are further supported by illustrative examples that confirm the validity and applicability of the results.

Keywords: Stability theory, Fractional calculus, Laplace transform, Gamma and beta functions, Integral transforms.

2010 MSC: 34K20, 26A33, 26D10, 44A10, 44A20, 33B15, 33B20.

### 1. Introduction

In mathematical analysis, differentiation and integration are typically considered in the context of integer-order operations. However, when these operations are generalized to fractional, irrational, or complex orders, the resulting field is known as fractional calculus. The historical development of fractional calculus can be traced as far back as the 17th century. In 1695, Leibniz questioned the meaning of derivatives of non-integer order in his correspondence with L'Hôpital. This idea was later developed through the contributions of eminent mathematicians such as Euler, Fourier, Liouville, and Riemann. For detailed information on these and the other developments, see, for example [1, 2, 4, 3, 5, 6] and references therein.

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Research on fractional differential equations (FDEs) has significantly expanded with the development of various definitions of fractional derivatives, leading to a rich and comprehensive analytical framework. Several types of fractional derivatives have been proposed, including those of Riemann–Liouville, Caputo, Hadamard, Hilfer, Grünwald–Letnikov, Riesz, Katugampola, Caputo–Fabrizio, and Atangana–Baleanu–Caputo. These have found extensive applications in modeling problems in physics and engineering [3, 4, 5, 6, 7, 8, 9]. However, as most of these definitions do not fully coincide with each other, there remains a need for more generalized formulations of fractional derivatives.

In this direction, Sousa and Oliveira [10] introduced the  $\psi$ -Hilfer fractional derivative ( $\psi$ -HFD), which is defined with respect to another function and has been successfully applied in the qualitative analysis of both linear and nonlinear FDEs see for example [11, 12, 13, 14]. The existence and uniqueness of some problems formulated using the  $\psi$ -Hilfer fractional derivative defined by Sousa and Oliveira have been analyzed in detail by various researchers. For example, Almalahi et al. [15] analyzed the existence and uniqueness of a new class of terminal type boundary value problems for  $\psi$ -Hilfer fractional differential equations using the Banach contraction principle and the Krasnosleskii fixed-point theorem. Wahash et al. [16] using the Picard iterative method, investigated the existence and uniqueness of a class of Cauchy problems for differential equations involving Caputo fractional derivatives with respect to another function. Similarly, Almalahi et al. [17] investigated existence and uniqueness results for fractional functional differential equations with boundary conditions and time lags containing Hilfer-type derivatives, and Shatanawi et al. [18] conducted an existence and uniqueness investigation using the Krasnosleskii fixed-point theorem for fractional operators with a special model kernel.

In 2007, Díaz and Pariguan [19] proposed the definitions of the  $k$ -gamma and  $k$ -beta functions such as

$$\Gamma_k(u) = \int_0^{\infty} z^{u-1} e^{-\frac{z^k}{k}} dz, \quad \beta_k(u, v) = \frac{1}{k} \int_0^1 z^{\frac{u}{k}-1} (1-z)^{\frac{v}{k}-1} dz, \quad \operatorname{Re} u > 0, \operatorname{Re} v > 0, k > 0,$$

which is the generalization of the Euler gamma and Euler beta functions  $\Gamma(\cdot)$  and  $\beta(\cdot, \cdot)$ , respectively and satisfies the following properties:

a)  $\lim_{k \rightarrow 1} \Gamma_k(u) = \Gamma(u)$ ,  $\Gamma_k(k) = 1$ ,  $\Gamma_k(u+k) = u\Gamma_k(u)$ ,  $\Gamma_k(u) = k^{\frac{u}{k}-1} \Gamma\left(\frac{u}{k}\right)$  and  $\Gamma(u+1) = u!$  for  $u \in \mathbb{N}$ ,

b)  $\lim_{k \rightarrow 1} \beta_k(u, v) = \beta(u, v)$ ,  $\beta_k(u, v) = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}$  and  $\beta_k(u, v) = \frac{1}{k} \beta\left(\frac{u}{k}, \frac{v}{k}\right)$

Although many fractional operators have since been defined using these functions, a comprehensive operator encompassing all of them has long been lacking.

In response, Kucche and Mali [11] proposed the generalized  $(k, \psi)$ -Hilfer fractional derivative ( $(k, \psi)$ -HFD), and later, Salim et al. [20] further extended it to define the tempered  $(k, \psi)$ -HFD. These advancements reflect the ongoing evolution and broadening of fractional calculus theory.

Within the framework of FDEs, Hyers-Ulam stability occupies a special place. In 1940, Stanislaw Ulam questioned the proximity of approximate solutions of functional equations to exact solutions, and in 1941, Donald Hyers provided an answer for linear functional equations [21]. Later, in 1978, Themistocles Rassias generalized this result to include

nonlinear equations. The concept, now known as Hyers-Ulam-Rassias stability, has also been employed in the analysis of FDEs [22, 23, 24, 25, 26, 27, 28, 29, 30].

While classical Laplace transforms were traditionally used in stability analyses of such FDEs [31, 32, 33, 34, 35, 36, 37, 38, 39], the increasing variety of generalized fractional differential and integral operators over time has rendered the classical Laplace transform insufficient for investigating the Hyers-Ulam stability of these equations.

The Ulam-type stability of generalized fractional differential equations (GFDEs) has been studied using methods such as successive approximations [40, 24, 41], fixed point theorems [40, 24, 11, 12, 14, 42, 43, 30], and Gronwall-type inequalities [44, 45]. Additionally, the  $\rho$ -Laplace transform, introduced by Abdeljawad [46] and later extended by Jarad and collaborators [47, 48], has led to new results concerning the solution of GFDEs [49, 50, 51].

More recently, Başıcı et al. [52] proposed the  $(k, \psi)$ -Generalized Laplace Transform ( $(k, \psi)$ -GLT), which provides a broad generalization of the classical Laplace transform and  $\rho$ -Laplace transform. Mısıır et al. [53], showed that this transform is not only an extension of the  $\rho$ -Laplace transform but also a generalization of many Laplace-type transforms introduced in the literature and has a significant advantage in obtaining solutions of fractional differential equations. This new transform allows for a more holistic and flexible analytical approach to the solutions of various FDEs. Moreover, Mısıır et al. [54] demonstrated the Hyers-Ulam stability (HUS) of FDEs involving the  $\psi$ -Riemann–Liouville fractional derivative by using the  $(k, \psi)$ -GLT. However, studies on this transformation remain relatively limited in number.

For this reason we establish in this paper Hyers-Ulam stability results to the initial value problem of the following tempered  $(k, \psi)$ - Hilfer fractional differential equations  $((k, \psi)$ -HFDEs) of the forms

$$\left( {}^{\text{TH}}\mathcal{D}_{k, a+}^{\alpha, \nu, \lambda; \psi} \mathbf{y} \right) (t) = f(t), \quad (1.1)$$

$$\left( {}^{\text{TH}}\mathcal{D}_{k, a+}^{\alpha, \nu, \lambda; \psi} \mathbf{y} \right) (t) - \eta \mathbf{y}(t) = f(t), \quad (1.2)$$

with the initial conditions

$$\Omega_{1, \psi}^{\lambda, m} \left( {}^{\text{T}}\mathcal{I}_{k, a+}^{(1-\nu)(kn-\alpha); \psi} \mathbf{y} \right) (a) = c_m, \quad m = 0, 1, 2, \dots, n-1 \quad (1.3)$$

by using the  $(k, \psi)$ -GLT, where  $0 < T < +\infty$ ,  $\nu \in [0, 1]$ ,  $\eta, c_m$  are scalars in  $\mathbb{C}$ ,  $n-1 < \frac{\alpha}{k} < n$ ,  $\lambda \in \mathbb{R}$ ,  $f \in C((0, T) \times \mathbb{C})$ ,  $\psi \in C^n[a, b]$  such that  $\psi'(t) > 0$  on  $[a, b]$  and  ${}^{\text{TH}}\mathcal{D}_{k, a+}^{\alpha, \nu; \psi}$  is the tempered  $(k, \psi)$ -HFD of order  $\alpha > 0$ . The notation  $\Omega_{k, \psi}^{\lambda, m} \left( \mathcal{I}_{k, a+}^{\gamma; \psi} \mathbf{y} \right)$ ,  $\gamma > 0$  will be presented in the next section.

The primary objective of this paper is to investigate the Hyers-Ulam stability (HUS) for Hilfer fractional differential equations (HFDEs) (1.1) and (1.2) when  $\alpha > 0$ , utilizing the  $(k, \psi)$ -GLT.

The structure of this article is as follows: Section 2, starts by fundamental definitions and properties are presented, along with background information regarding the tempered  $(k, \psi)$ -HFDs and the  $(k, \psi)$ -GLT. In the Section 3, using the lemmas and theorems we have established, the stability of problems (1.1) and (1.2) with the initial condition (1.3) is

investigated through the  $(k, \psi)$ -GLT, and the obtained analytical solutions are supported by relevant examples. Finally, the conclusion section highlights the advantages of analyzing the HUS of tempered  $(k, \psi)$ -HFDEs by means of the  $(k, \psi)$ -GLT.

## 2. Preliminaries and basic notations

In this section, we will introduce some basic definitions, notations, lemmas, and theorems that we will use throughout the paper. To simplify the notation and the prove of some results, we will introduce the  $\Omega_{k,\psi}^{\lambda,n} = \left(\frac{k}{\psi'(t)} \frac{d}{dt} + \lambda\right)^n$  and  $g_{k,\psi}^{[n]} = \left(\frac{k}{\psi'(t)} \frac{d}{dt}\right)^n g$  for  $k > 0, \lambda \in \mathbb{R}, n \in \mathbb{N}$  and  $\psi \in C^n [a, b]$  such that  $\psi'(t) > 0$  on  $[a, b]$ . Also, let's show,  $\psi_y(x) = \psi(x) - \psi(y)$  and  $\mathcal{U}_{\alpha,\beta}^{k,\lambda}(\psi_y(x)) = e^{-\lambda\psi_y(x)} (\psi_y(x))^{\frac{\beta}{k}-1} E_{\frac{\alpha}{k}, \frac{\beta}{k}}\left(\eta k^{-\frac{\alpha}{k}} (\psi_y(x))^{\frac{\alpha}{k}}\right)$  for  $\alpha > 0, \beta > 0$  and  $\lambda \in \mathbb{R}$ . Where  $\eta$  is a given constant.

**Definition 2.1.** [52] Let  $\psi \in C^n [a, b]$  such that  $\psi'(t) > 0$  on  $[a, b]$ . Then for  $n \in \mathbb{N}$

$$AC_{\psi}^n [a, b] = \left\{ g : (a, b] \rightarrow \mathbb{R} \text{ and } g_{k,\psi}^{[n-1]} \in AC [a, b] \right\}.$$

**Definition 2.2.** [52] Let  $I = [a, b]$  ( $0 < a < b < \infty$ ) be a finite interval. Also, let  $\psi : I \rightarrow \mathbb{R}$  and  $\psi'(t) > 0$  for all  $t \in I$ . Then for  $0 \leq \sigma < 1$ , the space  $C_{\sigma;\psi} (I, \mathbb{R})$  of weighted functions  $g$  defined on  $I$  as following:

$$C_{\sigma;\psi} [a, b] = \{g : (a, b] \rightarrow \mathbb{R} \text{ and } (\psi(\cdot) - \psi(a))^{\sigma} g(\cdot) \in C [a, b]\}$$

and

$$C_{\sigma;\psi}^n [a, b] = \left\{ g : g_{k,\psi}^{[n-1]} \in C [a, b] \text{ and } g_{k,\psi}^{[n]} \in C_{\sigma;\psi} [a, b] \right\}, n \in \mathbb{N},$$

where  $C_{0;\psi} (I, \mathbb{R}) = C [a, b]$  and  $C_{0;\psi}^n [a, b] = C^n [a, b]$ .

**Definition 2.3.** [20] ( $k$ -generalized  $\psi$ -fractional integral) Let  $g \in L^1 [a, b]$  and  $[a, b]$  be a finite or infinite interval on the real axis  $\mathbb{R}$ ,  $\psi(t) > 0$  be an increasing function on  $(a, b]$  and  $\psi'(t) > 0$  be continuous on  $(a, b)$ ,  $k > 0$  and  $\alpha > 0$ . Then the generalized  $k$ -fractional integral operators of a function  $g$  of order  $\alpha$  are defined by

$$I_{k,a+}^{\alpha;\psi} g(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t (\psi_u(t))^{\frac{\alpha}{k}-1} \psi'(u) g(u) du \tag{2.1}$$

and

$$I_{k,b-}^{\alpha;\psi} g(t) = \frac{1}{k\Gamma_k(\alpha)} \int_t^b (\psi_t(u))^{\frac{\alpha}{k}-1} \psi'(u) g(u) du. \tag{2.2}$$

**Theorem 2.4.** [11] Let  $\alpha > 0, \beta > 0$  and  $k > 0$ . Then we have the following semigroup property given by

$$I_{k,a+}^{\alpha;\psi} I_{k,a+}^{\beta;\lambda;\psi} g(t) = I_{k,a+}^{\alpha+\beta;\psi} g(t) = I_{k,a+}^{\beta;\psi} I_{k,a+}^{\alpha;\psi} g(t)$$

and

$$I_{k,b-}^{\alpha;\psi} I_{k,b-}^{\beta;\psi} g(t) = I_{k,b-}^{\alpha+\beta;\psi} g(t) = I_{k,b-}^{\beta;\psi} I_{k,b-}^{\alpha;\psi} g(t).$$

**Definition 2.5.** [11] Let  $k, \alpha > 0$ ,  $n - 1 < \frac{\alpha}{k} < n$  and  $\nu \in [0, 1]$ . Also, let  $\psi \in C^n [a, b]$ , ( $n \in \mathbb{N}$ )  $\psi'(t) > 0$  on  $(a, b)$  and  $g \in C^n [a, b]$ . Then the  $(k, \psi)$ -HFD of function  $g$  with respect to another function  $\psi$  on  $[a, b]$  of order  $\alpha$  is defined by

$${}^H D_{k,a+}^{\alpha,\nu;\psi} g(t) = I_{k,a+}^{\nu(kn-\alpha);\psi} \left( \frac{k}{\psi'(t)} \frac{d}{dt} \right)^n I_{k,a+}^{(1-\nu)(kn-\alpha);\psi} g(t). \quad (2.3)$$

In particular, if we take  $\nu = 0$  in (2.3)  $(k, \psi)$ -HFD reduces to  $(k, \psi)$ -Riemann Liouville fractional derivative operator and we can write as below:

$${}^H D_{k,a+}^{\alpha;\psi} g(t) = \frac{\left( \frac{k}{\psi'(t)} \frac{d}{dt} \right)^n}{k\Gamma_k(kn-\alpha)} \int_a^t (\psi_u(t))^{\frac{\alpha}{k}-1} g(u) \psi'(u) du. \quad (2.4)$$

**Theorem 2.6.** [11] Let  $\alpha, k \in \mathbb{R}^+$  and let  $\beta \in \mathbb{R}$  such that  $\frac{\beta}{k} > -1$ . Then

$${}^H D_{k,a+}^{\alpha,\nu;\psi} \left( (\psi_a(t))^{\frac{\beta}{k}} \right) = \frac{\Gamma_k(\beta+k)}{\Gamma_k(\beta+k-\alpha)} (\psi_a(t))^{\frac{\beta-\alpha}{k}}. \quad (2.5)$$

**Definition 2.7.** [20] (The  $(k, \psi)$ -tempered fractional integral) Let  $g \in L^1 [a, b]$  and  $[a, b]$  be a finite or infinite interval on the real axis  $\mathbb{R}$ ,  $\psi(t) > 0$  be an increasing function on  $(a, b)$  and be continuous on  $(a, b)$ ,  $\lambda \in \mathbb{R}$ ,  $k > 0$  and  $\alpha > 0$ . The  $(k, \psi)$ -tempered fractional integral operators of a function  $g$  of order  $\alpha$  and index  $\lambda$  are defined by

$$\begin{aligned} {}^T I_{k,a+}^{\alpha,\lambda;\psi} g(t) &= e^{-\lambda\psi(t)} I_{k,a+}^{\alpha;\psi} \left( g(t) e^{\lambda\psi(t)} \right) \\ &= \frac{1}{k\Gamma_k(\alpha)} \int_a^t (\psi_u(t))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_u(t)} \psi'(u) g(u) du \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} {}^T I_{k,b-}^{\alpha,\lambda;\psi} g(t) &= e^{\lambda\psi(t)} I_{k,b-}^{\alpha;\psi} \left( g(t) e^{-\lambda\psi(t)} \right) \\ &= \frac{1}{k\Gamma_k(\alpha)} \int_t^b (\psi_t(u))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_t(u)} \psi'(u) g(u) du \end{aligned} \quad (2.7)$$

where  $\Gamma_k(\cdot)$  is  $k$ -gamma function. If we take  $k = 1$  in the  $(k, \psi)$ -tempered fractional integral  ${}^T I_{k,a+}^{\alpha,\lambda;\psi}$  reduces to the  $\psi$ -tempered fractional integral  ${}^T I_{k,a+}^{\alpha;\psi}$  in [20].

**Theorem 2.8.** [20] Let  $\alpha > 0$ ,  $\beta > 0$  and  $k > 0$ . Then we have the following semigroup property given by

$${}^T I_{k,a+}^{\alpha,\lambda;\psi} {}^T I_{k,a+}^{\beta,\lambda;\psi} g(t) = {}^T I_{k,a+}^{\alpha+\beta,\lambda;\psi} g(t) = {}^T I_{k,a+}^{\beta,\lambda;\psi} {}^T I_{k,a+}^{\alpha,\lambda;\psi} g(t)$$

and

$${}^T I_{k,b-}^{\alpha,\lambda;\psi} {}^T I_{k,b-}^{\beta,\lambda;\psi} g(t) = {}^T I_{k,b-}^{\alpha+\beta,\lambda;\psi} g(t) = {}^T I_{k,b-}^{\beta,\lambda;\psi} {}^T I_{k,b-}^{\alpha,\lambda;\psi} g(t).$$

**Definition 2.9.** [20] (The tempered  $(k, \psi)$  –HFD) Let  $n - 1 < \frac{\alpha}{k} < n$  with  $n \in \mathbb{N}, \lambda \in \mathbb{R}, k > 0$ , an interval such that  $-\infty \leq a < b \leq \infty$  and  $g, \psi \in C^n([a, b], \mathbb{R})$  two functions such that  $\psi$  increasing and  $\psi'(t) \neq 0$ , for all  $t \in [a, b]$ . Then the tempered  $(k, \psi)$ -HFDs (left-side and right-side)  ${}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi}(\cdot)$  and  ${}^{\text{TH}}D_{k,b-}^{\alpha,\nu,\lambda;\psi}(\cdot)$  of a function  $g$  of order  $\alpha$  and index  $\lambda$  are type  $0 \leq \nu \leq 1$ , are defined by

$$\begin{aligned} {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} g(t) &= \left( {}^{\text{T}}I_{k,a+}^{\nu(kn-\alpha),\lambda;\psi} \left( \frac{1}{\psi'(t)} \frac{d}{dt} + \lambda \right)^n \left( k^n \left( {}^{\text{T}}I_{k,a+}^{(1-\nu)(kn-\alpha),\lambda;\psi} g \right) \right) \right) (t) \\ &= \left( {}^{\text{T}}I_{k,a+}^{\nu(kn-\alpha),\lambda;\psi} \Omega_{1,\psi}^{\lambda,n} \left( k^n \left( {}^{\text{T}}I_{k,a+}^{(1-\nu)(kn-\alpha),\lambda;\psi} g \right) \right) \right) (t) \\ &= e^{-\lambda\psi(t)} \left( {}^{\text{H}}D_{k,a+}^{\alpha,\nu;\psi} \left( g(t) e^{\lambda\psi(t)} \right) \right) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} {}^{\text{TH}}D_{k,b-}^{\alpha,\nu,\lambda;\psi} g(t) &= \left( {}^{\text{T}}I_{k,b-}^{\nu(kn-\alpha),\lambda;\psi} \left( -\frac{1}{\psi'(t)} \frac{d}{dt} + \lambda \right)^n \left( k^n \left( {}^{\text{T}}I_{k,b-}^{(1-\nu)(kn-\alpha),\lambda;\psi} g \right) \right) \right) (t) \\ &= \left( {}^{\text{T}}I_{k,b-}^{\nu(kn-\alpha),\lambda;\psi} (-1)^n \Omega_{1,\psi}^{\lambda,n} \left( k^n \left( {}^{\text{T}}I_{k,b-}^{(1-\nu)(kn-\alpha),\lambda;\psi} g \right) \right) \right) (t) \\ &= e^{\lambda\psi(t)} \left( {}^{\text{H}}D_{k,a+}^{\alpha,\nu;\psi} \left( g(t) e^{-\lambda\psi(t)} \right) \right). \end{aligned} \tag{2.9}$$

If we take  $k = 1$  in the tempered  $(k, \psi)$ -HFD  ${}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi}(\cdot)$  reduces to the tempered  $\psi$ -HFD  ${}^{\text{TH}}D_{a+}^{\alpha,\nu,\lambda;\psi}(\cdot)$  in [20].

**Definition 2.10.** [52] Let  $g, \psi : [a, \infty) \rightarrow \mathbb{R}$  be real valued functions such that  $\psi$  is continuous and  $\psi'(t) > 0$  on  $(a, b)$ . Also, let  $\rho, k > 0$ . Then, the  $(k, \psi)$ -GLT of  $g$  is defined by the improper integral

$$\mathcal{L}_{k,a+}^{\rho;\psi} \{g(t)\}(s) = \int_a^\infty e^{-s\psi_a(t)k^{1-\frac{\rho}{k}}} g(t)\psi'(t) dt. \tag{2.10}$$

provided that the integral in (2.10) exists, i.e., that the integral is convergent.

*Remark 2.11.* If we take  $\psi(t) = t, k = \rho$  and  $a = 0$  or  $\psi(t) = t, k = 1$  and  $a = 0$  in (2.10), we get the classical Laplace transform. If we take  $k = \rho$  and  $a = 0$  or  $k = 1$  and  $a = 0$  in (2.10), we get the  $\mathcal{L}_\rho \{g(t)\}(s)$  generalized Laplace transform given in [48]. If we take  $\psi(t) = \frac{t^\rho}{\rho}, k = \rho$  and  $a = 0$  or  $\psi(t) = \frac{t^\rho}{\rho}, k = 1$  and  $a = 0$  in (2.10), we get the  $\rho$ -generalized Laplace transform given in [51, 47].

**Theorem 2.12.** [52] Let  $g, \psi \in C[a, \infty)$  be real valued functions such that  $\psi$  is continuous and  $\psi'(t) > 0$  on  $[a, b)$ . Also, let  $\rho, k > 0$  and the  $(k, \psi)$ -GLT of  $g$  exists. Then

$$\mathcal{L}_{k,a+}^{\rho;\psi} \{g(t)\}(s) = \frac{1}{k^{1-\frac{\rho}{k}}} \mathcal{L} \left\{ g \left( \psi^{-1} \left( \frac{t}{k^{1-\frac{\rho}{k}}} + \psi(a) \right) \right) \right\} (s). \tag{2.11}$$

Note that  $\mathcal{L} \{g(t)\}(s)$  is the classical Laplace transform of  $g(t)$ .

**Definition 2.13.** [52] A function  $g$  defined on  $[0, \infty)$  taking values in  $\mathbb{R}$  is said to be of  $\psi$ -exponential order if one can find positive constants  $M, c, T$  such that the inequality  $|g(t)| \leq Me^{c\psi(t)}$  holds for all  $t \geq T$ .

**Theorem 2.14.** [52] If  $g : [a, \infty) \rightarrow \mathbb{R}$  is a piecewise continuous function and is  $\psi$ -exponential order, then its  $(k, \psi)$ -GLT exists for  $s > c$ .

**Theorem 2.15.** [52] If the  $(k, \psi)$ -GLT of  $g_1 : [a, \infty) \rightarrow \mathbb{R}$  exists for  $s > d_1$ , and the  $(k, \psi)$ -GLT of  $g_2 : [a, \infty) \rightarrow \mathbb{R}$  exists for  $s > d_2$ . Then for any constants  $c_1$  and  $c_2$ , the  $(k, \psi)$ -GLT of  $c_1g_1 + c_2g_2$  exist, then

$$\mathcal{L}_{k,a+}^{\rho;\psi} \{c_1g_1(t) + c_2g_2(t)\}(s) = c_1\mathcal{L}_{k,a+}^{\rho;\psi} \{g_1(t)\}(s) + c_2\mathcal{L}_{k,a+}^{\rho;\psi} \{g_2(t)\}(s) \quad (2.12)$$

for  $s > \max\{d_1, d_2\}$ .

**Definition 2.16.** [48] Consider two functions  $g$  and  $h$ , both of exponential order and piecewise continuous on any interval  $[a, T]$ . Their generalized convolution is defined by

$$(g *_{\psi} h)(t) = \int_a^t g(u) h(\psi^{-1}(\psi_u(t) + \psi(a))) \psi'(u) du. \quad (2.13)$$

The following lemma gives that  $g$  and  $h$  are commutative.

**Lemma 2.17.** [48] Consider two functions  $g$  and  $h$ , both of exponential order and piecewise continuous on any interval  $[a, T]$ . Then

$$g *_{\psi} h = h *_{\psi} g.$$

**Theorem 2.18.** [52] Consider two functions  $g$  and  $h$ , both of exponential order and piecewise continuous on any interval  $[a, T]$ . Then

$$\mathcal{L}_{k,a+}^{\rho;\psi} \{g *_{\psi} h\}(s) = \mathcal{L}_{k,a+}^{\rho;\psi} \{g\}(s) \mathcal{L}_{k,a+}^{\rho;\psi} \{h\}(s). \quad (2.14)$$

**Theorem 2.19.** [54] A function  $g(t) \in C_{\psi}[a, T]$  and of  $\psi$ -exponential order such that  $g_{1,\psi}^{[1]}$  is a piecewise continuous over every finite interval  $[a, T]$ . Then the  $(k, \psi)$ -GLT of  $g_{1,\psi}^{[1]}$  exist and

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ g_{1,\psi}^{[1]}(t) \right\}(s) = sk^{1-\frac{\rho}{k}} \mathcal{L}_{k,a+}^{\rho;\psi} \{g(t)\}(s) - g(a). \quad (2.15)$$

**Definition 2.20.** [55] Mittag-Leffler function of one parameter  $E_{\xi}(t)$  and two parameter  $E_{\xi,\eta}(t)$  are defined as

$$E_{\xi}(t) = \sum_{l=0}^{\infty} \frac{t^l}{\Gamma(\xi l + 1)} \text{ and } E_{\xi,\eta}(t) = \sum_{l=0}^{\infty} \frac{t^l}{\Gamma(\xi l + \eta)} \quad (2.16)$$

respectively. Where  $\text{Re}\xi > 0, \text{Re}\eta > 0$  and  $t \in \mathbb{C}$ . It is easy to show that  $E_1(t) = E_{1,1}(t) = e^t$ .

In the 19th century Prym [56] studied the theory of the incomplete gamma function for the case  $u = 1$ . A hypergeometric convergent function, which is a generalized form of the incomplete gamma functions, was systematically studied by Tricomi in 1950 [57].

**Definition 2.21.** The incomplete gamma functions  $\varphi(u, v)$  and  $\Gamma(u, v)$  are defined by

$$\varphi(u, v) = \int_0^v e^{-z} z^{u-1} dz, \quad \operatorname{Re}(u) > 0, v \geq 0 \quad (2.17)$$

and

$$\Gamma(u, v) = \int_v^\infty e^{-z} z^{u-1} dz, \quad \operatorname{Re}(u) > 0, v \geq 0. \quad (2.18)$$

These two functions satisfy the identity  $\varphi(u, v) + \Gamma(u, v) = \Gamma(u)$ , where  $\Gamma(u)$  denotes the classical gamma function.

Various properties and behaviors of the incomplete gamma function are collected, for example, in [56, 57, 58], where we mention only strongly relevant papers.

### 3. Main Results

The subsequent theorems are devoted to proving the HUS of equations (1.1) and (1.2), subject to initial conditions (1.3), using the  $(k, \psi)$ -GLT.

**Lemma 3.1.** Let  $[a, b]$  be a finite or infinite interval on the real axis  $\mathbb{R}$ ,  $\psi(t) > 0$  be an increasing function on  $(a, b]$  and be continuous on  $(a, b)$ ,  $\lambda > 0$ ,  $k > 0$  and  $\alpha > 0$ . Then

$$\begin{aligned} \int_a^t (\psi_u(t))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_u(t)} \psi'(u) du &= \int_a^t (\psi_\alpha(u))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_\alpha(u)} \psi'(u) du \\ &= \frac{1}{\lambda^{\frac{\alpha}{k}}} \left( \Gamma\left(\frac{\alpha}{k}\right) - \Gamma\left(\frac{\alpha}{k}, \lambda\psi_\alpha(t)\right) \right) \\ &= \frac{1}{\lambda^{\frac{\alpha}{k}}} \varphi\left(\frac{\alpha}{k}, \lambda\psi_\alpha(t)\right), \end{aligned} \quad (3.1)$$

where  $\Gamma\left(\frac{\alpha}{k}, \lambda\psi_\alpha(t)\right)$  and  $\varphi\left(\frac{\alpha}{k}, \lambda\psi_\alpha(t)\right)$  are incomplete gamma functions.

*Proof.* Following integral

$$\int_a^t (\psi_u(t))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_u(t)} \psi'(u) du$$

with change variable  $\xi = \lambda\psi_u(t)$ , and following integral

$$\int_a^t (\psi_\alpha(u))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_\alpha(u)} \psi'(u) du$$

with change variable  $\xi = \lambda\psi_a(u)$ , we can obtain

$$\begin{aligned} \int_a^t (\psi_u(t))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_u(t)} \psi'(u) du &= \int_a^t (\psi_a(u))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_a(u)} \psi'(u) du \\ &= \frac{1}{\lambda^{\frac{\alpha}{k}}} \int_0^{\lambda\psi_a(t)} e^{-\xi} \xi^{\frac{\alpha}{k}-1} d\xi. \end{aligned} \tag{3.2}$$

If we use gamma function and incomplete gamma function definition

$$\begin{aligned} \Gamma\left(\frac{\alpha}{k}\right) &= \int_0^\infty e^{-\xi} \xi^{\frac{\alpha}{k}-1} d\xi = \int_0^{\lambda\psi_a(t)} e^{-\xi} \xi^{\frac{\alpha}{k}-1} d\xi + \int_{\lambda\psi_a(t)}^\infty e^{-\xi} \xi^{\frac{\alpha}{k}-1} d\xi \\ &= \int_0^{\lambda\psi_a(t)} e^{-\xi} \xi^{\frac{\alpha}{k}-1} d\xi + \Gamma\left(\frac{\alpha}{k}, \lambda\psi_a(t)\right). \end{aligned}$$

Then we can write

$$\int_0^{\lambda\psi_a(t)} e^{-\xi} \xi^{\frac{\alpha}{k}-1} d\xi = \Gamma\left(\frac{\alpha}{k}\right) - \Gamma\left(\frac{\alpha}{k}, \lambda\psi_a(t)\right) = \varphi\left(\frac{\alpha}{k}, \lambda\psi_a(t)\right). \tag{3.3}$$

If we use (3.3) in (3.2), we can obtain

$$\begin{aligned} \int_a^t (\psi_u(t))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_u(t)} \psi'(u) du &= \int_a^t (\psi_a(u))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_a(u)} \psi'(u) du \\ &= \frac{1}{\lambda^{\frac{\alpha}{k}}} \left( \Gamma\left(\frac{\alpha}{k}\right) - \Gamma\left(\frac{\alpha}{k}, \lambda\psi_a(t)\right) \right) \\ &= \frac{1}{\lambda^{\frac{\alpha}{k}}} \varphi\left(\frac{\alpha}{k}, \lambda\psi_a(t)\right). \end{aligned}$$

□

**Lemma 3.2.** A function  $g(t) \in C_\psi[a, T]$  and of  $\psi$ -exponential order such that  $\Omega_{1,\psi}^{\lambda,1}$  is a piecewise continuous over every finite interval  $[a, T]$ . Then the  $(k, \psi)$ -GLT of  $\Omega_{1,\psi}^{\lambda,1}$  exists and

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \Omega_{1,\psi}^{\lambda,1} (g(t)) \right\} (s) = (sk^{1-\frac{\rho}{k}} + \lambda) \mathcal{L}_{k,a+}^{\rho;\psi} \{g(t)\} (s) - g(a). \tag{3.4}$$

*Proof.* If we use (2.10), we can write

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \Omega_{1,\psi}^{\lambda,1} (g(t)) \right\} (s) = \int_a^\infty e^{-s\psi_a(t)k^{1-\frac{\rho}{k}}} \Omega_{1,\psi}^{\lambda,1} (g(t)) \psi'(t) dt. \tag{3.5}$$

If use  $\Omega_{1,\psi}^{\lambda,1}(g(t)) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} + \lambda\right) g(t)$  in (3.5), we have

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \Omega_{1,\psi}^{\lambda,1}(g(t)) \right\} (s) &= \int_a^\infty e^{-s\psi_a(t)k^{1-\frac{\rho}{k}}} \left( \frac{1}{\psi'(t)} \frac{d}{dt} + \lambda \right) g(t) \psi'(t) dt \\ &= \int_a^\infty e^{-s\psi_a(t)k^{1-\frac{\rho}{k}}} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) g(t) \psi'(t) dt \\ &\quad + \lambda \int_a^\infty e^{-s\psi_a(t)k^{1-\frac{\rho}{k}}} g(t) \psi'(t) dt. \end{aligned} \tag{3.6}$$

For first integral, if we use (2.15) in (3.6), we can obtain

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \Omega_{1,\psi}^{\lambda,1}(g(t)) \right\} (s) = \left( sk^{1-\frac{\rho}{k}} + \lambda \right) \mathcal{L}_{k,a+}^{\rho;\psi} \{g(t)\} (s) - g(a).$$

□

Now we give generalize the Lemma 3.2.

**Corollary 3.3.** Let  $g(t) \in C^{n-1}[a, T]$  such that  $\Omega_{1,\psi}^{\lambda,j}$  ( $j = 0, 1, \dots, n-1$ ) are  $\psi$ -exponential order. Also, let  $\Omega_{1,\psi}^{\lambda,j}$  be a piecewise continuous over every finite interval  $[a, T]$ . Then the  $(k, \psi)$ -GLT of  $\Omega_{1,\psi}^{\lambda,n}$  exists and

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \Omega_{1,\psi}^{\lambda,n}(g(t)) \right\} (s) &= \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^n \mathcal{L}_{k,a+}^{\rho;\psi} \{g(t)\} (s) \\ &\quad - \sum_{j=0}^{n-1} \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{n-j-1} \Omega_{1,\psi}^{\lambda,j}(g(a)) \end{aligned} \tag{3.7}$$

*Proof.* The proof of the theorem is done by mathematical induction. □

**Lemma 3.4.** Let  $\psi \in C[a, \infty)$  be real valued functions such that  $\psi$  is continuous and  $\psi'(t) > 0$  on  $[a, b)$ . Also, let  $k > 0, \lambda \in \mathbb{R}$ . Then for  $s > 0$  and  $\text{Re}\beta > -1$

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ e^{-\lambda\psi_a(t)} (\psi_a(t))^\beta \right\} (s) = \frac{\Gamma(\beta + 1)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\beta+1}} = \frac{\Gamma_k((\beta + 1)k)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\beta+1} k^\beta}. \tag{3.8}$$

*Proof.* If we use  $(k, \psi)$ -GLT definition, we can write

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ e^{-\lambda\psi_a(t)} (\psi_a(t))^\beta \right\} (s) = \int_a^\infty e^{-(sk^{1-\frac{\rho}{k}} + \lambda)\psi_a(t)} (\psi_a(t))^\beta \psi'(t) dt$$

with change variable  $\left( sk^{1-\frac{\rho}{k}} + \lambda \right) \psi_a(t) = u$ , we have

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ e^{-\lambda\psi_a(t)} (\psi_a(t))^\beta \right\} (s) = \frac{1}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\beta+1}} \int_0^\infty e^{-u} u^\beta du.$$

If we use gamma function definition, we have

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ e^{-\lambda \psi_a(t)} (\psi_a(t))^\beta \right\} (s) = \frac{\Gamma(\beta + 1)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\beta+1}} = \frac{\Gamma_k((\beta + 1)k)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\beta+1} k^\beta}.$$

□

*Remark 3.5.* If we take  $\lambda = 0$  in Lemma 3.4, we can obtain

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ (\psi_a(t))^\beta \right\} (s) = \frac{\Gamma(\beta + 1)}{\left( sk^{1-\frac{\rho}{k}} \right)^{\beta+1}} = \frac{\Gamma_k((\beta + 1)k)}{\left( sk^{1-\frac{\rho}{k}} \right)^{\beta+1} k^\beta} \quad s > 0, \operatorname{Re}\beta > -1 \quad (3.9)$$

**Lemma 3.6.** Let  $\operatorname{Re}\alpha > 0, \operatorname{Re}\delta > 0$  and  $\left| \frac{\eta k^{-\frac{\alpha}{k}}}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}}} \right| < 1$ . Then

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \mathcal{U}_{\alpha,\delta}^{k,\lambda}(\psi_a(t)) \right\} (s) = \frac{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha-\delta}{k}}}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} - \eta k^{-\frac{\alpha}{k}}} \quad (3.10)$$

*Proof.* If we use (2.16), we can write

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \mathcal{U}_{\alpha,\delta}^{k,\lambda}(\psi_a(t)) \right\} (s) = \sum_{j=0}^{\infty} \frac{\left( \eta k^{-\frac{\alpha}{k}} \right)^j}{\Gamma\left(\frac{\alpha j + \delta}{k}\right)} \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ e^{-\lambda \psi_a(t)} (\psi_a(t))^{\frac{\alpha j + \delta}{k} - 1} \right\} (s).$$

If we use (3.8), for  $\beta = \frac{\alpha j + \delta}{k} - 1$ , we have

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \mathcal{U}_{\alpha,\delta}^{k,\lambda}(\psi_a(t)) \right\} (s) &= \frac{1}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\delta}{k}}} \sum_{j=0}^{\infty} \left( \frac{\eta k^{-\frac{\alpha}{k}}}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}}} \right)^j \\ &= \frac{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha-\delta}{k}}}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} - \eta k^{-\frac{\alpha}{k}}}. \end{aligned}$$

□

*Remark 3.7.* If we take  $\lambda = 0$  in Lemma 3.5, we get

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ (\psi_a(t))^{\frac{\delta}{k}-1} E_{\frac{\alpha}{k}, \frac{\delta}{k}} \left( \eta k^{-\frac{\alpha}{k}} (\psi_a(t))^{\frac{\alpha}{k}} \right) \right\} (s) = \frac{\left( sk^{1-\frac{\rho}{k}} \right)^{\frac{\alpha-\delta}{k}}}{\left( sk^{1-\frac{\rho}{k}} \right)^{\frac{\alpha}{k}} - \eta k^{-\frac{\alpha}{k}}}. \quad (3.11)$$

*Remark 3.8.* If we take  $\alpha = \delta$  in Lemma 3.5, we get

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \mathcal{U}_{\alpha,\alpha}^{k,\lambda}(\psi_a(t)) \right\} (s) = \frac{1}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} - \eta k^{-\frac{\alpha}{k}}}. \quad (3.12)$$

*Remark 3.9.* If we take  $\lambda = 0$  and  $\alpha = \delta$  in Lemma 3.5, we get

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ (\psi_a(t))^{\frac{\alpha}{k}-1} E_{\frac{\alpha}{k}, \frac{\alpha}{k}} \left( \eta k^{-\frac{\alpha}{k}} (\psi_a(t))^{\frac{\alpha}{k}} \right) \right\} (s) = \frac{1}{\left( sk^{1-\frac{\rho}{k}} \right)^{\frac{\alpha}{k}} - \eta k^{-\frac{\alpha}{k}}}. \quad (3.13)$$

**Lemma 3.10.** Let  $g(t)$  be a piecewise continuous over every finite interval  $[a, T]$  and  $\psi$ -exponential order. Also, let  $\alpha > 0$  and  $\psi'(t) > 0$ . Then

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ {}^T I_{k,a+}^{\alpha,\lambda;\psi} g(t) \right\} (s) = \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{g(t)\} (s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} k^{\frac{\alpha}{k}}}. \quad (3.14)$$

*Proof.* If we use (2.6), we have

$${}^T I_{k,a+}^{\alpha,\lambda;\psi} g(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t (\psi_u(t))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_u(t)} \psi'(u) g(u) du$$

with a change variables  $\tau = \psi^{-1}(\psi_u(t) + \psi(a))$ , we get

$${}^T I_{k,a+}^{\alpha,\lambda;\psi} g(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t (\psi_a(\tau))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_a(\tau)} g(\psi^{-1}(\psi_\tau(t) + \psi(a))) \psi'(\tau) d\tau. \quad (3.15)$$

If we take  $(k, \psi)$ -GLT both side of (3.15) and use (2.13), (2.14), we have

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ {}^T I_{k,a+}^{\alpha,\lambda;\psi} g(t) \right\} (s) &= \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \frac{1}{k\Gamma_k(\alpha)} \right. \\ &\quad \left. \times \int_a^t (\psi_a(\tau))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_a(\tau)} g(\psi^{-1}(\psi_\tau(t) + \psi(a))) \psi'(\tau) d\tau \right\} (s) \\ &= \frac{1}{k\Gamma_k(\alpha)} \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ (\psi_a(t))^{\frac{\alpha}{k}-1} e^{-\lambda\psi_a(t)} \right\} (s) \mathcal{L}_{k,a+}^{\rho;\psi} \{g(t)\} (s). \end{aligned}$$

If we use (3.8) in last equation, because of  $\frac{\alpha}{k} - 1 > -1$ , we can obtain,

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ {}^T I_{k,a+}^{\alpha,\lambda;\psi} g(t) \right\} (s) &= \frac{1}{k\Gamma_k(\alpha)} \frac{\Gamma_k(\alpha)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} k^{\frac{\alpha}{k}-1}} \mathcal{L}_{k,a+}^{\rho;\psi} \{g(t)\} (s) \\ &= \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{g(t)\} (s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} k^{\frac{\alpha}{k}}}. \end{aligned}$$

□

**Lemma 3.11.** Let  $\alpha > 0$  and  $g \in AC_{\psi}^n[a, b]$  for any  $b > a$ ,  $\psi \in C_{\sigma;\psi}^n[a, b]$  such that  $\psi'(t) > 0$  and  $\Omega_{k,\psi}^{\lambda,j} \left( I_{k,a+}^{(1-\nu)(kn-\alpha)-k;j,\lambda;\psi} g \right)$ ,  $j = 0, 1, \dots, n-1$  be of  $\psi$ -exponential order.

Then

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ {}^{\text{TH}}\mathcal{D}_{k,a+}^{\alpha,\nu,\lambda;\psi} g(t) \right\} (s) &= k^n \left[ \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^n \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{g(t)\} (s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{k n - \alpha}{k}} k^{\frac{k n - \alpha}{k}}} \right. \\ &\quad - \sum_{j=0}^{n-1} \frac{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{n-j-1}}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(k n - \alpha)}{k}} k^{\frac{\nu(k n - \alpha)}{k}}} \\ &\quad \left. \times \Omega_{1,\psi}^{\lambda,j} \left( \mathbb{T}_{k,a+}^{(1-\nu)(k n - \alpha);\psi} g \right) (a) \right]. \end{aligned} \tag{3.16}$$

*Proof.* If we use the  $(k, \psi)$ -GLT defined by (2.10), we obtain

$$\begin{aligned} &\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ {}^{\text{TH}}\mathcal{D}_{k,a+}^{\alpha,\nu,\lambda;\psi} g(t) \right\} (s) \\ &= \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \mathbb{T}_{k,a+}^{\nu(k n - \alpha),\lambda;\psi} \left( \frac{1}{\psi'(t)} \frac{d}{dt} + \lambda \right)^n \left( k^n \left( \mathbb{T}_{k,a+}^{(1-\nu)(k n - \alpha),\lambda;\psi} \right) g(t) \right) \right\} (s) \\ &= \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \mathbb{I}_{k,a+}^{\nu(k n - \alpha),\lambda;\psi} \Omega_{1,\psi}^{\lambda,n} \left( k^n \left( \mathbb{T}_{k,a+}^{(1-\nu)(k n - \alpha),\lambda;\psi} \right) g(t) \right) \right\} (s). \end{aligned} \tag{3.17}$$

If we use (3.7) and (3.14) in (3.17), we have

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ {}^{\text{TH}}\mathcal{D}_{k,a+}^{\alpha,\nu,\lambda;\psi} g(t) \right\} (s) &= \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \Omega_{1,\psi}^{\lambda,n} \left( k^n \left( \mathbb{T}_{k,a+}^{(1-\nu)(k n - \alpha),\lambda;\psi} \right) g(t) \right) \right\} (s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(k n - \alpha)}{k}} k^{\frac{\nu(k n - \alpha)}{k}}} \\ &= \frac{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^n \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \left( k^n \left( \mathbb{T}_{k,a+}^{(1-\nu)(k n - \alpha),\lambda;\psi} \right) g(t) \right) \right\} (s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(k n - \alpha)}{k}} k^{\frac{\nu(k n - \alpha)}{k}}} \\ &= \frac{\sum_{j=0}^{n-1} \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{n-j-1} \left( \Omega_{1,\psi}^{\lambda,j} \left( k^n \left( \mathbb{T}_{k,a+}^{(1-\nu)(k n - \alpha),\lambda;\psi} \right) g \right) (a) \right)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(k n - \alpha)}{k}} k^{\frac{\nu(k n - \alpha)}{k}}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{k^n \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^n}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(kn-\alpha)}{k}} k^{\frac{\nu(kn-\alpha)}{k}}} \\
 &\times \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{g(t)\}(s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{(1-\nu)(kn-\alpha)}{k}} k^{\frac{(1-\nu)(kn-\alpha)}{k}}} \\
 &\frac{\sum_{j=0}^{n-1} \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{n-j-1} \left( \Omega_{1,\psi}^{\lambda,j} \left( k^n \left( \mathbb{T} \mathbb{I}_{k,a+}^{(1-\nu)(kn-\alpha),\lambda;\psi} \right) g \right) (a) \right)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(kn-\alpha)}{k}} k^{\frac{\nu(kn-\alpha)}{k}}} \\
 &= k^n \left[ \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^n \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{g(t)\}(s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{kn-\alpha}{k}} k^{\frac{kn-\alpha}{k}}} \right. \\
 &\quad \left. - \sum_{j=0}^{n-1} \frac{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{n-j-1} \left( \Omega_{1,\psi}^{\lambda,j} \left( \mathbb{T} \mathbb{I}_{k,a+}^{(1-\nu)(kn-\alpha),\lambda;\psi} \right) g \right) (a)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(kn-\alpha)}{k}} k^{\frac{\nu(kn-\alpha)}{k}}} \right].
 \end{aligned}$$

□

**Lemma 3.12.** Let  $\alpha, k \in \mathbb{R}^+, \beta \in \mathbb{R}$  and  $\frac{\beta}{k} > 0$ . Then

$${}^{\text{TH}}\mathbb{D}_{k,a+}^{\alpha,\nu,\lambda;\psi} \left( e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{\beta}{k}-1} \right) = \frac{\Gamma_k(\beta)}{\Gamma_k(\beta-\alpha)} e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{\beta-\alpha}{k}-1}. \quad (3.18)$$

*Proof.* If we use tempered  $(k, \psi)$ -HFD definition in (2.8) for  $g(t) = e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{\beta}{k}-1}$ , we can write

$$\begin{aligned}
 {}^{\text{TH}}\mathbb{D}_{k,a+}^{\alpha,\nu,\lambda;\psi} \left( e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{\beta}{k}-1} \right) &= e^{-\lambda\psi(t)} \left( {}^{\text{H}}\mathbb{D}_{k,a+}^{\alpha,\nu;\psi} \left( e^{\lambda\psi(a)} (\psi_a(t))^{\frac{\beta}{k}-1} \right) \right) \\
 &= e^{-\lambda\psi_a(t)} \left( {}^{\text{H}}\mathbb{D}_{k,a+}^{\alpha,\nu;\psi} \left( (\psi_a(t))^{\frac{\beta}{k}-1} \right) \right) \quad (3.19)
 \end{aligned}$$

If we use (2.5), we can write,

$${}^{\text{H}}\mathbb{D}_{k,a+}^{\alpha,\nu;\psi} \left( (\psi_a(t))^{\frac{\beta}{k}-1} \right) = \frac{\Gamma_k(\beta)}{\Gamma_k(\beta-\alpha)} (\psi_a(t))^{\frac{\beta-\alpha}{k}-1}. \quad (3.20)$$

If we use (3.20) in (3.19), we can obtain

$${}^{\text{TH}}\mathbb{D}_{k,a+}^{\alpha,\nu,\lambda;\psi} \left( e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{\beta}{k}-1} \right) = \frac{\Gamma_k(\beta)}{\Gamma_k(\beta-\alpha)} e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{\beta-\alpha}{k}-1}.$$

□

**Definition 3.13.** We say that equation (1.1) possesses the HUS with initial conditions (1.3) if there exists a positive constant  $K > 0$  such that, for any given  $\varepsilon > 0$  and a function  $y$  satisfying  $\left| \left( {}^{\text{TH}}\mathbb{D}_{k,a+}^{\alpha,\nu,\lambda;\psi} y \right) (t) - f(t) \right| \leq \varepsilon$ , there exists a solution  $y_e$  of the differential equation (1.1) such that  $|y(t) - y_e(t)| \leq \varepsilon K$ .

**Definition 3.14.** We say that equation (1.2) possesses the HUS with initial conditions (1.3) if there exists a positive constant  $K > 0$  such that, for any given  $\varepsilon > 0$  and a function  $y$  satisfying  $\left| \left( {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} y \right) (t) - \lambda y(t) - f(t) \right| \leq \varepsilon$ , there exists a solution  $y_e$  of the differential equation (1.2) such that  $|y(t) - y_e(t)| \leq \varepsilon K$ .

**Theorem 3.15.** Let  $\alpha, k \in \mathbb{R}^+, 0 < T < \infty, \psi(t)$  be an increasing and positive function on  $(a, b], (-\infty \leq a < b \leq \infty)$ . If a function  $y : (0, T] \rightarrow \mathbb{C}$  satisfies the inequality

$$\left| \left( {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} y \right) (t) - f(t) \right| \leq \varepsilon \tag{3.21}$$

with the initial conditions (1.3) for each  $t \in (0, T]$  and some  $\varepsilon > 0$ , then there exists a solution  $y_e : (0, T] \rightarrow \mathbb{C}$  of the differential equation (1.1) such that

$$|y(t) - y_e(t)| \leq \frac{\varepsilon}{k\Gamma_k(\alpha) \lambda^{\frac{\alpha}{k}}} \varphi\left(\frac{\alpha}{k}, \lambda\psi_a(T)\right). \tag{3.22}$$

*Proof.* Let

$$Y_1(t) = \left( {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} y \right) (t) - f(t) \tag{3.23}$$

for  $t \in (0, T]$ . Taking the  $(k, \psi)$ -GLT both side of (3.23) and if we use (3.16) we have

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \{Y_1(t)\}(s) &= \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \left( {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} y \right) (t) \right\} (s) - \mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\}(s) \\ &= k^n \left[ \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^n \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{y(t)\}(s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{kn-\alpha}{k}} k^{\frac{kn-\alpha}{k}}} \right. \\ &\quad \left. - \sum_{j=0}^{n-1} \frac{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{n-j-1} \left( \Omega_{1,\psi}^{\lambda,j} \left( {}^{\text{T}}I_{k,a+}^{(1-\nu)(kn-\alpha),\lambda;\psi} y \right) (a) \right)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(kn-\alpha)}{k}} k^{\frac{\nu(kn-\alpha)}{k}}} \right] \\ &\quad - \mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\}(s). \end{aligned} \tag{3.24}$$

If we rewrite (3.24) and use the initial conditions (1.3), we obtain

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \{y(t)\}(s) &= \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\}(s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} k^{\frac{\alpha}{k}}} + k^n \sum_{j=0}^{n-1} \frac{c_j \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{n-j-1}}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(kn-\alpha)+\alpha}{k}} k^{\frac{\nu(kn-\alpha)+\alpha}{k}}} \\ &\quad + \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{Y_1(t)\}(s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} k^{\frac{\alpha}{k}}}. \end{aligned} \tag{3.25}$$

By (3.8) for  $\beta = \frac{\alpha}{k} - 1$  and  $\beta = \frac{\nu(kn-\alpha)+\alpha}{k} - n + j$  ( $j = 0, 1, 2, \dots, n - 1$ ) respectively, we

have

$$\left. \begin{aligned} \frac{1}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} k^{\frac{\alpha}{k}}} &= \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ e^{-\lambda \psi_a(t)} (\psi_a(t))^{\frac{\alpha}{k}-1} \right\} (s)}{k \Gamma_k(\alpha)}, \\ \frac{1}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(kn-\alpha)+\alpha}{k} - n + j + 1} k^{\frac{\nu(kn-\alpha)+\alpha}{k} - n + j}} &= \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ e^{-\lambda \psi_a(t)} (\psi_a(t))^{\frac{\nu(kn-\alpha)+\alpha}{k} - n + j} \right\} (s)}{\Gamma_k \left( \left( \frac{\nu(kn-\alpha)+\alpha}{k} - n + j + 1 \right) k \right)}. \end{aligned} \right\} \quad (3.26)$$

If we use (3.26) in (3.25), we get

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \{y(t)\} (s) &= \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\} (s) \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ e^{-\lambda \psi_a(t)} (\psi_a(t))^{\frac{\alpha}{k}-1} \right\} (s)}{k \Gamma_k(\alpha)} \\ &+ \sum_{j=0}^{n-1} \frac{c_j k^j \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ e^{-\lambda \psi_a(t)} (\psi_a(t))^{\frac{\nu(kn-\alpha)+\alpha}{k} + j - n} \right\} (s)}{\Gamma_k \left( \left( \frac{\nu(kn-\alpha)+\alpha}{k} + j - n + 1 \right) k \right)} \\ &+ \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{Y_1(t)\} (s) \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ e^{-\lambda \psi_a(t)} (\psi_a(t))^{\frac{\alpha}{k}-1} \right\} (s)}{k \Gamma_k(\alpha)}. \end{aligned} \quad (3.27)$$

Setting

$$y_e(t) = \frac{e^{-\lambda \psi_a(t)} (\psi_a(t))^{\frac{\alpha}{k}-1}}{k \Gamma_k(\alpha)} *_{\psi} f(t) + \sum_{j=0}^{n-1} \frac{k^j c_j e^{-\lambda \psi_a(t)} (\psi_a(t))^{\frac{\nu(kn-\alpha)+\alpha}{k} + j - n}}{\Gamma_k \left( \left( \frac{\nu(kn-\alpha)+\alpha}{k} + j - n + 1 \right) k \right)}. \quad (3.28)$$

Taking the  $(k, \psi)$ -GLT both side of (3.28) and if we use (2.14), (3.26), one get

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \{y_e(t)\} (s) &= \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \frac{e^{-\lambda \psi_a(t)} (\psi_a(t))^{\frac{\alpha}{k}-1}}{k \Gamma_k(\alpha)} \right\} (s) \mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\} (s) \\ &+ \sum_{j=0}^{n-1} \frac{k^j c_j \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ e^{-\lambda \psi_a(t)} (\psi_a(t))^{\frac{\nu(kn-\alpha)+\alpha}{k} + j - n} \right\} (s)}{\Gamma_k \left( \left( \frac{\nu(kn-\alpha)+\alpha}{k} + j - n + 1 \right) k \right)} \\ &= \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\} (s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} k^{\frac{\alpha}{k}}} \\ &+ \sum_{j=0}^{n-1} \frac{c_j}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(kn-\alpha)+\alpha}{k} - n + j + 1} k^{\frac{\nu(kn-\alpha)+\alpha}{k} - n}}. \end{aligned} \quad (3.29)$$

By (1.3), (3.16), and (3.29), we get

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} y_e(t) \right\} (s) &= k^n \left[ \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^n \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{y_e(t)\}(s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{k n - \alpha}{k}} k^{\frac{k n - \alpha}{k}}} \right. \\ &\quad \left. - \sum_{j=0}^{n-1} \frac{c_j \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{n-j-1}}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(k n - \alpha)}{k}} k^{\frac{\nu(k n - \alpha)}{k}}} \right] \\ &= \frac{1}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{-\frac{\alpha}{k}} k^{-\frac{\alpha}{k}}} \left\{ \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\}(s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} k^{\frac{\alpha}{k}}} \right. \\ &\quad \left. + \sum_{j=0}^{n-1} \frac{c_j}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(k n - \alpha) + \alpha}{k} - n + j + 1} k^{\frac{\nu(k n - \alpha) + \alpha}{k} - n}} \right\} \\ &\quad - k^n \sum_{j=0}^{n-1} \frac{c_j \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{n-j-1}}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(k n - \alpha)}{k}} k^{\frac{\nu(k n - \alpha)}{k}}} \\ &= \mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\}(s). \end{aligned}$$

Due to the fact that  $\mathcal{L}_{k,a+}^{\rho;\psi}$  is one-to-one infers that

$$\left( {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} y_e \right) (t) = f(t).$$

So,  $y_e(t)$  is a solution of equation (1.1). If we use (3.27) and (3.29), we get

$$\mathcal{L}_{k,a+}^{\rho;\psi} \{y(t) - y_e(t)\}(s) = \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ Y_1(t) *_{\psi} \frac{e^{-\lambda \psi_a(t)} (\psi_a(t))^{\frac{\alpha}{k}-1}}{k \Gamma_k(\alpha)} \right\} (s).$$

If we use again the fact  $\mathcal{L}_{k,a+}^{\rho;\psi}$  is one-to-one, we get

$$y(t) - y_e(t) = \frac{1}{k \Gamma_k(\alpha)} \left[ Y_1(t) *_{\psi} e^{-\lambda \psi_a(t)} (\psi_a(t))^{\frac{\alpha}{k}-1} \right].$$

Taking the absolute value of both sides of the last equation and if we use (2.13) and (3.21), it follows that

$$\begin{aligned} |y(t) - y_e(t)| &\leq \frac{1}{k \Gamma_k(\alpha)} \left| \int_a^t Y_1(u) e^{-\lambda \psi_u(t)} (\psi_u(t))^{\frac{\alpha}{k}-1} \psi'(u) du \right| \\ &= \frac{1}{k \Gamma_k(\alpha)} \int_a^t |Y_1(u)| \left| e^{-\lambda \psi_u(t)} (\psi_u(t))^{\frac{\alpha}{k}-1} \psi'(u) \right| du \\ &\leq \frac{\varepsilon}{k \Gamma_k(\alpha)} \int_a^t \left| e^{-\lambda \psi_u(t)} (\psi_u(t))^{\frac{\alpha}{k}-1} \psi'(u) \right| du. \end{aligned}$$

If we use (3.1), we have

$$\frac{1}{k\Gamma_k(\alpha)} \int_a^t \left| e^{-\lambda\psi_u(t)} (\psi_u(t))^{\frac{\alpha}{k}-1} \psi'(u) \right| du = \frac{1}{k\Gamma_k(\alpha)\lambda^{\frac{\alpha}{k}}} \left( \Gamma\left(\frac{\alpha}{k}\right) - \Gamma\left(\frac{\alpha}{k}, \lambda\psi_a(t)\right) \right).$$

Then we can obtain

$$\begin{aligned} |y(t) - y_e(t)| &\leq \frac{\varepsilon}{k\Gamma_k(\alpha)\lambda^{\frac{\alpha}{k}}} \left( \Gamma\left(\frac{\alpha}{k}\right) - \Gamma\left(\frac{\alpha}{k}, \lambda\psi_a(t)\right) \right) \\ &< \frac{\varepsilon}{k\Gamma_k(\alpha)\lambda^{\frac{\alpha}{k}}} \left( \Gamma\left(\frac{\alpha}{k}\right) - \Gamma\left(\frac{\alpha}{k}, \lambda\psi_a(T)\right) \right) \\ &= \frac{\varepsilon}{k\Gamma_k(\alpha)\lambda^{\frac{\alpha}{k}}} \varphi\left(\frac{\alpha}{k}, \lambda\psi_a(T)\right). \end{aligned}$$

which completes the proof. □

*Remark 3.16.* If  $T < \infty$ , then because of  $\frac{\alpha}{k} > 0$  the term  $\varphi\left(\frac{\alpha}{k}, \lambda\psi_a(T)\right)$  is bounded. Therefore, for the constant  $K = \frac{\varphi\left(\frac{\alpha}{k}, \lambda\psi_a(T)\right)}{k\Gamma_k(\alpha)\lambda^{\frac{\alpha}{k}}}$ , the fractional differential equation (1.1) is Hyers–Ulam stable. Note that, (1.1) is not Hyers–Ulam stable at  $T = \infty$ .

*Remark 3.17.* If we take  $k = 1, \lambda = 0$  and  $\nu = 0$  in  ${}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi}$  it reduces to the fractional derivative  $D_{a+}^{\alpha;\psi}$ . Therefore the results obtained in Theorem 3.11 include the results of Theorem 3.8 in [54].

**Theorem 3.18.** Let  $\alpha, k \in \mathbb{R}^+, 0 < T < \infty, \eta$  be a scalar,  $\psi(t)$  be an increasing and positive function on  $(a, b]$  ( $-\infty \leq a < b < \infty$ ) and  $f(t)$  be a given real continuous function on  $[0, \infty)$ . If a function  $y : (0, T] \rightarrow \mathbb{C}$  satisfies the following inequality

$$\left| \left( {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} y \right) (t) - \eta y(t) - f(t) \right| \leq \varepsilon \tag{3.30}$$

with the initial conditions (1.3) for each  $t \in (0, T]$  and some  $\varepsilon > 0$ , then there exists a solution  $y_e : (0, T] \rightarrow \mathbb{C}$  of (1.2) such that

$$|y(t) - y_e(t)| = \varepsilon \sum_{m=0}^{\infty} \frac{|\eta k^{-\frac{\alpha}{k}}|^m}{\Gamma\left(\frac{\alpha m + \alpha}{k}\right) (k\lambda^{m+1})^{\frac{\alpha}{k}}} \varphi\left(\frac{\alpha m + \alpha}{k}, \lambda\psi_a(T)\right). \tag{3.31}$$

*Proof.* Let

$$Y_2(t) = \left( {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} y \right) (t) - \eta y(t) - f(t), \tag{3.32}$$

for  $t \in (0, T]$ . Taking the  $(k, \psi)$ -GLT of (3.32) and if we use (3.16) we have

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \{Y_2(t)\}(s) &= \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} y(t) \right\}(s) - \eta \mathcal{L}_{k,a+}^{\rho;\psi} \{y(t)\}(s) - \mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\}(s) \\ &= k^n \left[ \frac{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^n \mathcal{L}_{k,a+}^{\rho;\psi} \{y(t)\}(s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{kn-\alpha}{k}} k^{\frac{kn-\alpha}{k}}} \right. \\ &\quad \left. - \sum_{j=0}^{n-1} \frac{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{n-j-1} \left( \Omega_{1,\psi}^{\lambda,j} \left( {}^{\text{T}}I_{k,a+}^{(1-\nu)(kn-\alpha),\lambda;\psi} y \right) (a) \right)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(kn-\alpha)}{k}} k^{\frac{\nu(kn-\alpha)}{k}}} \right] \\ &\quad - \eta \mathcal{L}_{k,a+}^{\rho;\psi} \{y(t)\}(s) - \mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\}(s). \end{aligned} \tag{3.33}$$

If we write  $\beta = \alpha + \nu(kn - \alpha) - k(n - j - 1)$  and use the initial conditions (1.3), we can write

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \{y(t)\}(s) &= \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{f(t)\}(s)}{k^{\frac{\alpha}{k}} \left( \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} - \eta k^{-\frac{\alpha}{k}} \right)} + \sum_{j=0}^{n-1} \frac{k^{n-\frac{\nu(kn-\alpha)}{k}} c_j \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha-\beta}{k}}}{k^{\frac{\alpha}{k}} \left( \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} - \eta k^{-\frac{\alpha}{k}} \right)} \\ &\quad + \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{Y_2(t)\}(s)}{k^{\frac{\alpha}{k}} \left( \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} - \eta k^{-\frac{\alpha}{k}} \right)}. \end{aligned}$$

By using (2.14), (3.10) and (3.12), we can write

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \{y(t)\}(s) &= \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \frac{f(t)}{k^{\frac{\alpha}{k}}} *_{\psi} \mathcal{U}_{\alpha,\alpha}^{k,\lambda}(\psi_a(t)) \right\}(s) \\ &\quad + \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \sum_{j=0}^{n-1} k^{n-\frac{\nu(kn-\alpha)+\alpha}{k}} c_j \mathcal{U}_{\alpha,\beta}^{k,\lambda}(\psi_a(t)) \right\}(s) \\ &\quad + \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \frac{Y_2(t)}{k^{\frac{\alpha}{k}}} *_{\psi} \mathcal{U}_{\alpha,\alpha}^{k,\lambda}(\psi_a(t)) \right\}(s). \end{aligned} \tag{3.34}$$

Setting

$$y_e(t) = \frac{f(t)}{k^{\frac{\alpha}{k}}} *_{\psi} \mathcal{U}_{\alpha,\alpha}^{k,\lambda}(\psi_a(t)) + \sum_{j=0}^{n-1} k^{n-\frac{\nu(kn-\alpha)+\alpha}{k}} c_j \mathcal{U}_{\alpha,\beta}^{k,\lambda}(\psi_a(t)). \tag{3.35}$$

Taking the  $(k, \psi)$  –GLT both side of (3.35), one has

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \{y_e(t)\}(s) &= \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \frac{f(t)}{k^{\frac{\alpha}{k}}} \right\}(s) \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \mathcal{U}_{\alpha,\alpha}^{k,\lambda}(\psi_a(t)) \right\}(s) \\ &\quad + \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \sum_{j=0}^{n-1} k^{n-\frac{\nu(kn-\alpha)+\alpha}{k}} c_j \mathcal{U}_{\alpha,\beta}^{k,\lambda}(\psi_a(t)) \right\}(s) \end{aligned} \tag{3.36}$$

By using the (3.16) and (1.3), we get

$$\begin{aligned} \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} y_e(t) - \eta y_e(t) \right\} (s) &= \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} y_e(t) \right\} (s) - \eta \mathcal{L}_{k,a+}^{\rho;\psi} \{ y_e(t) \} (s) \\ &= k^n \left[ \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^n \frac{\mathcal{L}_{k,a+}^{\rho;\psi} \{ y_e(t) \} (s)}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{kn-\alpha}{k}} k^{\frac{kn-\alpha}{k}}} \right. \\ &\quad \left. - \sum_{j=0}^{n-1} \frac{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{n-j-1} c_j}{\left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\nu(kn-\alpha)}{k}} k^{\frac{\nu(kn-\alpha)}{k}}} \right] \\ &\quad - \eta \mathcal{L}_{k,a+}^{\rho;\psi} \{ y_e(t) \} (s) \\ &= \left[ \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{\frac{\alpha}{k}} k^{\frac{\alpha}{k}} - \eta \right] \mathcal{L}_{k,a+}^{\rho;\psi} \{ y_e(t) \} (s) \\ &\quad - \sum_{j=0}^{n-1} k^{n-\frac{\nu(kn-\alpha)}{k}} \left( sk^{1-\frac{\rho}{k}} + \lambda \right)^{n-\frac{\nu(kn-\alpha)}{k}-j-1} c_j. \end{aligned}$$

If we use (3.36), (3.10) and (3.12) for  $\beta = \alpha + \nu(kn - \alpha) - k(n - j - 1)$  on the right side of last equation, we get

$$\mathcal{L}_{k,a+}^{\rho;\psi} \left\{ {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} y_e(t) - \eta y_e(t) \right\} (s) = \mathcal{L}_{k,a+}^{\rho;\psi} \{ f(t) \} (s).$$

Due to the fact that  $\mathcal{L}_{k,a+}^{\rho;\psi}$  is one-to-one infers that

$$\left( {}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi} y_e \right) (t) - \eta y_e(t) = f(t).$$

So,  $y_e(t)$  is a solution of equation (1.2). By using (3.34) and (3.36), we have

$$\mathcal{L}_{k,a+}^{\rho;\psi} \{ y(t) - y_e(t) \} (s) = \mathcal{L}_{k,a+}^{\rho;\psi} \left\{ \frac{Y_2(t)}{k^{\frac{\alpha}{k}}} *_{\psi} \mathcal{U}_{\alpha,\alpha}^{k,\lambda}(\psi_a(t)) \right\} (s).$$

If we use again the fact that  $\mathcal{L}_{k,a+}^{\rho;\psi}$  is one-to-one, we get

$$y(t) - y_e(t) = \frac{Y_2(t) *_{\psi} \mathcal{U}_{\alpha,\alpha}^{k,\lambda}(\psi_a(t))}{k^{\frac{\alpha}{k}}}. \tag{3.37}$$

If we use the generalized convolution definition given by (2.13) for  $\mathcal{U}_{\alpha,\alpha}^{k,\lambda}(\psi_a(t)) = e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{\alpha}{k}-1} E_{\frac{\alpha}{k},\frac{\alpha}{k}} \left( \eta k^{-\frac{\alpha}{k}} (\psi_a(t))^{\frac{\alpha}{k}} \right)$  and take the absolute value of both sides of (3.37), we get

$$\begin{aligned} |y(t) - y_e(t)| &= \frac{1}{k^{\frac{\alpha}{k}}} \left| \int_a^t Y_2(u) e^{-\lambda\psi_u(t)} (\psi_u(t))^{\frac{\alpha}{k}-1} E_{\frac{\alpha}{k},\frac{\alpha}{k}} \left( \eta k^{-\frac{\alpha}{k}} (\psi_u(t))^{\frac{\alpha}{k}} \right) \psi'(u) du \right| \\ &\leq \frac{1}{k^{\frac{\alpha}{k}}} \int_a^t |Y_2(u)| \left| e^{-\lambda\psi_u(t)} (\psi_u(t))^{\frac{\alpha}{k}-1} E_{\frac{\alpha}{k},\frac{\alpha}{k}} \left( \eta k^{-\frac{\alpha}{k}} (\psi_u(t))^{\frac{\alpha}{k}} \right) \psi'(u) \right| du. \end{aligned}$$

Therefore, from (3.30), it follows that

$$\begin{aligned} |y(t) - y_e(t)| &\leq \frac{\varepsilon}{k^{\frac{\alpha}{k}}} \int_a^t \left| e^{-\lambda\psi_u(t)} (\psi_u(t))^{\frac{\alpha}{k}-1} \sum_{m=0}^{\infty} \frac{(\eta k^{-\frac{\alpha}{k}} (\psi_u(t))^{\frac{\alpha}{k}})^m}{\Gamma(\frac{\alpha}{k}m + \frac{\alpha}{k})} \psi'(u) \right| du \\ &= \frac{\varepsilon}{k^{\frac{\alpha}{k}}} \sum_{m=0}^{\infty} \frac{|\eta k^{-\frac{\alpha}{k}}|^m}{\Gamma(\frac{\alpha m + \alpha}{k})} \int_a^t \left| e^{-\lambda\psi_u(t)} (\psi_u(t))^{\frac{\alpha}{k}m + \frac{\alpha}{k}-1} \psi'(u) \right| du \end{aligned}$$

For  $\alpha \rightarrow \frac{\alpha}{k}m + \frac{\alpha}{k}$  in (3.1), we have

$$\int_a^t \left| e^{-\lambda\psi_u(t)} (\psi_u(t))^{\frac{\alpha}{k}m + \frac{\alpha}{k}-1} \psi'(u) \right| du \leq \frac{1}{\lambda^{\frac{\alpha m + \alpha}{k}}} \left( \Gamma\left(\frac{\alpha m + \alpha}{k}\right) - \Gamma\left(\frac{\alpha m + \alpha}{k}, \lambda\psi_a(t)\right) \right).$$

Then

$$\begin{aligned} |y(t) - y_e(t)| &\leq \varepsilon \sum_{m=0}^{\infty} \frac{|\eta k^{-\frac{\alpha}{k}}|^m}{\Gamma(\frac{\alpha m + \alpha}{k}) (k\lambda^{m+1})^{\frac{\alpha}{k}}} \left( \Gamma\left(\frac{\alpha m + \alpha}{k}\right) - \Gamma\left(\frac{\alpha m + \alpha}{k}, \lambda\psi_a(t)\right) \right) \\ &< \varepsilon \sum_{m=0}^{\infty} \frac{|\eta k^{-\frac{\alpha}{k}}|^m}{\Gamma(\frac{\alpha m + \alpha}{k}) (k\lambda^{m+1})^{\frac{\alpha}{k}}} \left( \Gamma\left(\frac{\alpha m + \alpha}{k}\right) - \Gamma\left(\frac{\alpha m + \alpha}{k}, \lambda\psi_a(T)\right) \right) \\ &= \varepsilon \sum_{m=0}^{\infty} \frac{|\eta k^{-\frac{\alpha}{k}}|^m}{\Gamma(\frac{\alpha m + \alpha}{k}) (k\lambda^{m+1})^{\frac{\alpha}{k}}} \varphi\left(\frac{\alpha m + \alpha}{k}, \lambda\psi_a(T)\right) \end{aligned}$$

which completes the proof. □

*Remark 3.19.* If  $T < \infty$ , then because of  $\alpha, k \in \mathbb{R}^+$ , the term  $\varphi(\frac{\alpha m + \alpha}{k}, \lambda\psi_a(T))$  is bounded. Therefore, the fractional differential equation (1.2) is Hyers–Ulam stable for the constant  $K = \sum_{m=0}^{\infty} \frac{|\eta k^{-\frac{\alpha}{k}}|^m}{\Gamma(\frac{\alpha m + \alpha}{k}) (k\lambda^{m+1})^{\frac{\alpha}{k}}} \varphi\left(\frac{\alpha m + \alpha}{k}, \lambda\psi_a(T)\right)$ . Note that, (1.2) is not Hyers–Ulam stable at  $T = \infty$ .

*Remark 3.20.* If we take  $k = 1, \lambda = 0$  and  $\nu = 0$  in  ${}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi}$  it reduces to the fractional derivative  $D_{a+}^{\alpha;\psi}$ . Therefore the results obtained in Theorem 3.12 include the results of Theorem 3.10 in [54].

**Example 3.21.** Consider the following initial value problem:

$${}^{\text{TH}}D_{k,a+}^{\frac{1}{2},\nu,\lambda;\psi} y(t) = f(t), \quad \Omega_{1,\psi}^{\lambda,0} \left( I_{k,a+}^{(1-\nu)(k-\frac{1}{2});\psi} y \right) (a) = 0, \tag{3.38}$$

where  $\alpha = \frac{1}{2}$  and  $f(t) = \frac{\Gamma_k(4)}{\Gamma_k(\frac{7}{2})} (\psi_a(t))^{\frac{7}{2k}-1} e^{-\lambda\psi_a(t)} + \frac{1}{20}$ . For  $\frac{1}{20} < \varepsilon \leq 1$ , we show that by using the (3.18) the function  $y_1(t) = e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{4}{k}-1}$  satisfies

$$\begin{aligned} \left| {}^{\text{TH}}D_{k,a+}^{\frac{1}{2},\nu,\lambda;\psi} y_1(t) - f(t) \right| &= \left| \frac{\Gamma_k(4)}{\Gamma_k(\frac{7}{2})} (\psi_a(t))^{\frac{7}{2k}-1} e^{-\lambda\psi_a(t)} \right. \\ &\quad \left. - \frac{\Gamma_k(4)}{\Gamma_k(\frac{7}{2})} (\psi_a(t))^{\frac{7}{2k}-1} e^{-\lambda\psi_a(t)} - \frac{1}{20} \right| \\ &= \left| \frac{1}{20} \right| < \varepsilon. \end{aligned}$$

Also, the initial value of  $y_1(t)$  is  $\Omega_{1,\psi}^{\lambda,0} \left( I_{k,a+}^{(1-\nu)(k-\frac{1}{2});\psi} y_1 \right) (a) = 0$ . On the other hand, if we use the fact  $f(t) = \frac{\Gamma_k(4)}{\Gamma_k(\frac{7}{2})} (\psi_a(t))^{\frac{7}{2k}-1} e^{-\lambda\psi_a(t)} + \frac{1}{20}$  and  $\alpha = \frac{1}{2}$ ,  $n = 1$ ,  $c_0 = 0$  and (2.13) in (3.28), we get

$$\begin{aligned} y_e(t) &= \frac{1}{k\Gamma_k(\frac{1}{2})} \left( (\psi_a(t))^{\frac{1}{2k}-1} e^{-\lambda\psi_a(t)} *_{\psi} \left( \frac{\Gamma_k(4)}{\Gamma_k(\frac{7}{2})} (\psi_a(t))^{\frac{7}{2k}-1} e^{-\lambda\psi_a(t)} + \frac{1}{20} \right) \right) \\ &= \frac{1}{k\Gamma_k(\frac{1}{2})} \int_a^t (\psi_a(u))^{\frac{1}{2k}-1} e^{-\lambda\psi_a(u)} \left[ \frac{\Gamma_k(4)}{\Gamma_k(\frac{7}{2})} (\psi_u(t))^{\frac{7}{2k}-1} e^{-\lambda\psi_u(t)} + \frac{1}{20} \right] \psi'(u) du \\ &= \frac{\Gamma_k(4) e^{-\lambda\psi_a(t)}}{\Gamma_k(\frac{1}{2}) \Gamma_k(\frac{7}{2})} \frac{1}{k} \int_a^t (\psi_a(u))^{\frac{1}{2k}-1} (\psi_u(t))^{\frac{7}{2k}-1} \psi'(u) du \\ &\quad + \frac{1}{20k\Gamma_k(\frac{1}{2})} \int_a^t (\psi_a(u))^{\frac{1}{2k}-1} e^{-\lambda\psi_a(u)} \psi'(u) du. \end{aligned}$$

If we use with change variable  $u = \psi^{-1}(\psi(t) - \xi(\psi_a(t)))$  for first integral and (3.1) for second integral in last equation, we obtain

$$y_e(t) = e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{4}{k}-1} + \frac{1}{20k\Gamma_k(\frac{1}{2})} \frac{1}{\lambda^{\frac{1}{2k}}} \left( \Gamma\left(\frac{1}{2k}\right) - \Gamma\left(\frac{1}{2k}, \lambda\psi_a(t)\right) \right)$$

Then

$$\begin{aligned} |y_1(t) - y_e(t)| &= \left| e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{4}{k}-1} \right. \\ &\quad \left. - \left( e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{4}{k}-1} + \frac{1}{20k\Gamma_k(\frac{1}{2})} \frac{1}{\lambda^{\frac{1}{2k}}} \left( \Gamma\left(\frac{1}{2k}\right) - \Gamma\left(\frac{1}{2k}, \lambda\psi_a(t)\right) \right) \right) \right| \\ &= \left| \frac{1}{20k\Gamma_k(\frac{1}{2})} \frac{1}{\lambda^{\frac{1}{2k}}} \left( \Gamma\left(\frac{1}{2k}\right) - \Gamma\left(\frac{1}{2k}, \lambda\psi_a(t)\right) \right) \right| \\ &< \frac{\varepsilon}{k\Gamma_k(\frac{1}{2}) \lambda^{\frac{1}{2k}}} \left| \left( \Gamma\left(\frac{1}{2k}\right) - \Gamma\left(\frac{1}{2k}, \lambda\psi_a(t)\right) \right) \right| \\ &< \frac{\varepsilon}{k\Gamma_k(\frac{1}{2}) \lambda^{\frac{1}{2k}}} \left| \left( \Gamma\left(\frac{1}{2k}\right) - \Gamma\left(\frac{1}{2k}, \lambda\psi_a(T)\right) \right) \right| \\ &= \frac{\varepsilon}{k\Gamma_k(\frac{1}{2}) \lambda^{\frac{1}{2k}}} \varphi\left(\frac{1}{2k}, \lambda\psi_a(T)\right). \end{aligned}$$

Therefore (3.38) satisfies Theorem 3.11 for  $K = \frac{1}{k\Gamma_k(\frac{1}{2}) \lambda^{\frac{1}{2k}}} \varphi\left(\frac{1}{2k}, \lambda\psi_a(T)\right)$  and  $\varepsilon = \frac{1}{20}$ . Thus (3.38) is Hyers-Ulam stable.

*Remark 3.22.* If we take  $k = 1$ ,  $\lambda = 0$  and  $\nu = 0$  in  ${}^{TH}D_{k,a+}^{\alpha,\nu,\lambda;\psi}$  it reduces to the fractional derivative  $D_{a+}^{\alpha;\psi}$ . Therefore the results obtained in Example 3.13 include the results of Example 3.14 in [54].

**Example 3.23.** Consider the following initial value problem:

$${}^{\text{TH}}D_{k, a+}^{\frac{1}{2}, \nu, \lambda; \psi} y(t) + 7y(t) = f(t), \quad \Omega_{1, \psi}^{\lambda, 0} \left( I_{k, a+}^{(1-\nu)(k-\frac{1}{2}); \psi} y \right) (a) = 0, \quad (3.39)$$

where  $\lambda = -7$ ,  $\alpha = \frac{1}{2}$  and  $f(t) = \frac{\Gamma_k(4)}{\Gamma_k(\frac{7}{2})} (\psi_a(t))^{\frac{7}{2k}-1} e^{-\lambda\psi_a(t)} + 7(\psi_a(t))^{\frac{4}{k}-1} e^{-\lambda\psi_a(t)} + \frac{1}{20}$ . For  $\frac{1}{20} < \varepsilon \leq 1$  we show that by using (3.18) the function  $y_1(t) = e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{4}{k}-1}$  satisfies

$$\begin{aligned} \left| {}^{\text{TH}}D_{k, a+}^{\frac{1}{2}, \nu, \lambda; \psi} y_1(t) + 7y_1(t) - f(t) \right| &= \left| \frac{\Gamma_k(4)}{\Gamma_k(\frac{7}{2})} e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{7}{2k}-1} + 7e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{4}{k}-1} \right. \\ &\quad \left. - \frac{\Gamma_k(4)}{\Gamma_k(\frac{7}{2})} e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{7}{2k}-1} - 7e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{4}{k}-1} \right. \\ &\quad \left. - \frac{1}{20} \right| = \left| \frac{1}{20} \right| < \varepsilon. \end{aligned}$$

Also, the initial value of  $y_1(t)$  is  $\Omega_{1, \psi}^{\lambda, 0} \left( I_{k, a+}^{(1-\nu)(k-\frac{1}{2}); \psi} y_1 \right) (a) = 0$ . Because of  $\alpha = \frac{1}{2}$ ,  $n = 1$ ,  $c_0 = 0$  if we using (2.13),  $f(t) = \frac{\Gamma_k(4)}{\Gamma_k(\frac{7}{2})} (\psi_a(t))^{\frac{7}{2k}-1} e^{-\lambda\psi_a(t)} + 7(\psi_a(t))^{\frac{4}{k}-1} e^{-\lambda\psi_a(t)} + \frac{1}{20}$  in (3.35) and we have

$$\begin{aligned} y_e(t) &= \frac{1}{k^{\frac{1}{2k}}} \left( \frac{\Gamma_k(4)}{\Gamma_k(\frac{7}{2})} e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{7}{2k}-1} + 7e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{4}{k}-1} + \frac{1}{20} \right) \\ &\quad *_{\psi} \left( e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{1}{2k}-1} E_{\frac{1}{2k}, \frac{1}{2k}} \left( (-7) k^{-\frac{1}{2k}} (\psi_a(t))^{\frac{1}{2k}} \right) \right) \\ &= \frac{\Gamma_k(4) e^{-\lambda\psi_a(t)}}{\Gamma_k(\frac{7}{2}) k^{\frac{1}{2k}}} \int_a^t (\psi_a(u))^{\frac{7}{2k}-1} (\psi_u(t))^{\frac{1}{2k}-1} E_{\frac{1}{2k}, \frac{1}{2k}} \left( (-7) k^{-\frac{1}{2k}} (\psi_u(t))^{\frac{1}{2k}} \right) \psi'(u) du \\ &\quad + \frac{7e^{-\lambda\psi_a(t)}}{k^{\frac{1}{2k}}} \int_a^t (\psi_a(u))^{\frac{4}{2k}-1} (\psi_u(t))^{\frac{1}{2k}-1} E_{\frac{1}{2k}, \frac{1}{2k}} \left( (-7) k^{-\frac{1}{2k}} (\psi_u(t))^{\frac{1}{2k}} \right) \psi'(u) du \\ &\quad + \frac{1}{20k^{\frac{1}{2k}}} \int_a^t e^{-\lambda\psi_u(t)} (\psi_u(t))^{\frac{1}{2k}-1} E_{\frac{1}{2k}, \frac{1}{2k}} \left( (-7) k^{-\frac{1}{2k}} (\psi_u(t))^{\frac{1}{2k}} \right) \psi'(u) du. \end{aligned}$$

If we use change variable  $u = \psi^{-1}(\psi(t) - \xi(\psi_a(t)))$  for first integral and second integral and (3.1) for third integral in last equation, we have

$$y_e(t) = e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{4}{k}-1} + \frac{1}{20k^{\frac{1}{2k}}} \sum_{m=0}^{\infty} \frac{(-7k^{-\frac{1}{2k}})^m}{\lambda^{\frac{m+1}{2k}} \Gamma(\frac{m+1}{2k})} \left( \Gamma\left(\frac{m+1}{2k}\right) - \Gamma\left(\frac{m+1}{2k}, \lambda\psi_a(t)\right) \right).$$

Then

$$\begin{aligned}
 |y_1(t) - y_e(t)| &= \left| e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{4}{k}-1} - e^{-\lambda\psi_a(t)} (\psi_a(t))^{\frac{4}{k}-1} \right. \\
 &\quad \left. - \frac{1}{20k^{\frac{1}{2k}}} \sum_{m=0}^{\infty} \frac{(-7k^{-\frac{1}{2k}})^m}{\lambda^{\frac{m+1}{2k}} \Gamma(\frac{m+1}{2k})} \left( \Gamma\left(\frac{m+1}{2k}\right) - \Gamma\left(\frac{m+1}{2k}, \lambda\psi_a(t)\right) \right) \right| \\
 &= \frac{1}{20k^{\frac{1}{2k}}} \sum_{m=0}^{\infty} \frac{|-7k^{-\frac{1}{2k}}|^m}{\lambda^{\frac{m+1}{2k}} \Gamma(\frac{m+1}{2k})} \left( \Gamma\left(\frac{m+1}{2k}\right) - \Gamma\left(\frac{m+1}{2k}, \lambda\psi_a(t)\right) \right) \\
 &< \frac{\varepsilon}{k^{\frac{1}{2k}}} \sum_{m=0}^{\infty} \frac{|-7k^{-\frac{1}{2k}}|^m}{\lambda^{\frac{m+1}{2k}} \Gamma(\frac{m+1}{2k})} \left( \Gamma\left(\frac{m+1}{2k}\right) - \Gamma\left(\frac{m+1}{2k}, \lambda\psi_a(T)\right) \right) \\
 &= \frac{\varepsilon}{k^{\frac{1}{2k}}} \sum_{m=0}^{\infty} \frac{|-7k^{-\frac{1}{2k}}|^m}{\lambda^{\frac{m+1}{2k}} \Gamma(\frac{m+1}{2k})} \varphi\left(\frac{m+1}{2k}, \lambda\psi_a(T)\right)
 \end{aligned}$$

Therefore equation (3.39) satisfies the conditions of Theorem 3.12 for  $\frac{1}{20} < \varepsilon \leq 1$  and the constant  $K = \frac{1}{k^{\frac{1}{2k}}} \sum_{m=0}^{\infty} \frac{|-7k^{-\frac{1}{2k}}|^m}{\lambda^{\frac{m+1}{2k}} \Gamma(\frac{m+1}{2k})} \varphi\left(\frac{m+1}{2k}, \lambda\psi_a(T)\right)$ . Thus (3.39) is Hyers–Ulam stable.

*Remark 3.24.* If we take  $k = 1$ ,  $\lambda = 0$  and  $\nu = 0$  in  ${}^{\text{TH}}D_{k,a+}^{\alpha,\nu,\lambda;\psi}$  it reduces to the fractional derivative  $D_{a+}^{\alpha;\psi}$ . Therefore the results obtained in Example 3.14 include the results of Example 3.15 in [54].

#### 4. Conclusion

This study investigates the HUS of tempered  $(k, \psi)$ -HFDEs using the  $(k, \psi)$ -GLT method, introduced as a generalization of Laplace-type operators. For this purpose, tempered integral and derivative operators and their properties are examined. Furthermore, the values of these operators under  $(k, \psi)$ -GLT are calculated. Specifically, we established sufficient conditions for the HUS of these equations through this approach. Furthermore, we proposed a new method for investigating the HUS of differential equations. We also showed that this study serves as a generalization of certain existing works in the literature, which we supported through illustrative examples. To the best of our knowledge, this is the first study to employ the  $(k, \psi)$ -GLT in proving the HUS of tempered  $(k, \psi)$ -HFDEs.

#### 5. Acknowledgement

The authors thank the referee for his careful reading of the paper and for the valuable suggestions which greatly improved this work.

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