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Fractional Corrected Simpson's Second Formula Type Inequalities via Extended s-Convexity

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Abstract

In this paper, we establish new fractional variants of the corrected Simpson's second formula type inequalities by leveraging the concept of extended s-convexity. To achieve this, we first derive a novel integral identity involving Riemann–Liouville fractional integrals, which serves as a fundamental auxiliary result. Building upon this identity, we obtain several inequalities for functions whose first-order derivatives satisfy the extended s-convexity condition on a given interval. Furthermore, we demonstrate the practical relevance of our theoretical findings by applying them to derive estimates for special means. These applications highlight the utility of our inequalities in numerical analysis and approximation theory.

Keywords: Riemann-Liouville fractional integrals, extended s-convex functions, corrected Simpson's second formula, Hólder inequality, power mean inequality.
2010 MSC: 26A51, 26D10, 26D15.

1. Introduction

The well-known Simpson-type inequality in the literature is given by the following theorem:

Theorem 1.1. *Let* $f : [a, b] \to \mathbb{R}$ *be a four-times continuously differentiable mapping on* (a, b). *Then the following inequality*

$$\left|\frac{1}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \leqslant \frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty$$

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holds, where $\left\|f^{(4)}\right\|_{\infty}=\sup\limits_{x\in(a,b)}\left|f^{(4)}(x)\right|$.

The extensive applications of fractional calculus in both pure and applied mathematics as well as other disciplines have made it a prominent issue in mathematical analysis and drawn the interest of several scholars. It does, in fact, offer a very useful tool for characterizing memory and inheritable qualities of different materials and procedures. Although there are several fractional operators in the literature, Liouville is frequently given credit for this kind of computation. Here we first recall the definition of Riemann-Liouville operator.

Definition 1.2 ([13]). Assume $\mu,\xi_1>0$, and let $f\in L^1[\xi_1,\xi_2]$. The Riemann-Liouville fractional integrals $I^{\mu}_{\zeta_1^+}f$ and $I^{\mu}_{\zeta_2^-}f$ are defined as:

$$I^{\mu}_{\xi_1^+}f(x)=\frac{1}{\Gamma(\mu)}\int\limits_{\xi_1}^{x}(x-\rho)^{\mu-1}f(\rho)d\rho,\quad x>\xi_1,$$

$$I^{\mu}_{\xi_2^-}f(x)=\frac{1}{\Gamma(\mu)}\int\limits_{x}^{\xi_2}(\rho-x)^{\mu-1}f(\rho)d\rho,\quad \xi_2>x,$$

respectively, where $\Gamma(\alpha)=\int\limits_{0}^{\infty}e^{-\rho}\rho^{\mu-1}d\rho$, is the gamma function and $I^{0}_{\xi_1^+}f(x)=I^{0}_{\xi_2^-}f(x)=f(x)$.

Integral inequalities are one of the fields that regularly use fractional operators to establish generalizations, refinements and improvements. This concept is closely related to convexity. Hence several articles dealing with convex integral inequalities and their fractional analogues. For further details, we refer readers to [1, 2, 6, 8, 9, 11, 12, 15, 14, 16, 17, 19, 21, 22, 25, 26, 27, 29, 37] and references therein.

The so-called Hermite-Hadamard inequality (see [7]) is a well-known inequality for the class of convex functions. It can be declared as follows: For each convex function f on the interval $[\xi_1,\xi_2]$ with $\xi_1<\xi_2$, we have

$$f\left(\frac{\xi_1+\xi_2}{2}\right)\leqslant\frac{1}{\xi_2-\xi_1}\int\limits_{\xi_1}^{\xi_2}f(x)\,dx\leqslant\frac{f(\xi_1)+f(\xi_2)}{2}. \tag{1.1}$$

We recall that a function $f:I\subset\mathbb{R}\to\mathbb{R}$ is convex, if for every $x,y\in I$ and all $\rho\in[0,1]$ (see [17]), we have

$$f(\rho x+(1-\rho)y)\leqslant\rho f(x)+(1-\rho)f(y).$$

The notion of convexity has been generalized in various ways. Among of these generalization is the extended s-convexity in the second sens, defined as follows:

A nonnegative function $f:I\subset[0,\infty)\to\mathbb{R}$ is said to be extended s-convex for some fixed $s\in[-1,1]$, if

$$f(\rho x+(1-\rho)y)\ \leqslant\rho^{s}f(x)+(1-\rho)^{s}f(y)$$

holds for all $x, y \in I$ and $\rho \in [0,1]$ (see [39]).

The above concept recapture the following classes of functions: P-functions, Q-functions, s-Q-functions, convex function and s-convex functions in the second sense. The study of such extended convexity classes remains an active area of research, as demonstrated by recent work on extended Hermite-Hadamard inequalities [18]. This motivates our present investigation into fractional Simpson-type inequalities under the same framework."

Dragomir and Fitzpatrick [4] proved the following variant of inequality (1.1) which holds for s-convex functions in the second sense. For every positive and s-convex function in the second sense f on the interval $[\xi_1, \xi_2]$ with $\xi_1 < \xi_2$, we have

$$2^{s-1} f\left(\frac{\xi_1+\xi_2}{2}\right) \leqslant \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} f(x)\,dx \leqslant \frac{f(\xi_1)+f(\xi_2)}{s+1}. \tag{1.2}$$

In [10], Franjić and Pečarić proposed the following closed-type quadrature formula, known as the corrected Simpson 3/8 formula:

$$\begin{aligned}\int_{\xi_1}^{\xi_2} f(u)\,du =& \frac{b-a}{80}\left(13f(\xi_1)+27f\left(\frac{2\xi_1+\xi_2}{3}\right)+27f\left(\frac{\xi_1+2\xi_2}{3}\right)+13f(\xi_2)\right)\\ &-\frac{(b-a)^2}{120}\big(f'(b)-f'(a)\big)+\frac{(b-a)^7}{2721600}f^{(6)}(\eta),\end{aligned}$$

where $\eta \in [a,b]$.

Here, the term "corrected" refers to the inclusion of a correction term involving the first derivative evaluated at the endpoints of the interval; it is shown in [10] that this yields better estimates than the classical Simpson 3/8 formula.

In this paper, we first prove a new identity involving Riemann-Liouville fractional integrals. By using this identity, we establish some new fractional corrected Simpson 3/8 type inequalities for functions whose first derivatives are s-convex in the second sense. The work is concluded by some applications to special means.

2. Preliminaries

Definition 2.1 ([6]). Suppose that $R(x), R(y) > 0$ where x, y are complex numbers. The Beta function is given by the following integral:

$$B(x,y) = \int_0^1 \rho^{x-1}(1-\rho)^{y-1}\,d\rho.$$

The definition of the incomplete Beta function is given by:

Definition 2.2 ([6]). Suppose that $\Re(x), \Re(y) > 0$ where x, y are complex numbers. The incomplete Beta function is given by the following integral:

$$B_{\xi_1}(x,y) = \int_0^{\xi_1} \rho^{x-1}(1-\rho)^{y-1}\,d\rho, \quad \xi_1 < 1.$$

Definition 2.3 ([6])**.** Suppose that $\Re(c) > \Re(\xi_2) > 0$ and $|z| < 1$. The hypergeometric function is given by the following integral:

$$ {}_2F_1(\xi_1, \xi_2, c, z) = \frac{1}{B(\xi_2, c-\xi_2)} \int_0^1 \rho^{\xi_2 - 1} (1-\rho)^{c-\xi_2-1} (1-z\rho)^{-\xi_1} d\rho, $$

where $B(.,.)$ is the Beta function.

3. Main results

In what follows, we consider the following notation:

$$ \mathcal{S}(f) = \left(I^{\alpha}_{\frac{2a+b}{3}^-} f(a) + I^{\alpha}_{\frac{a+2b}{3}^+} f(b) \right) + 2^{\alpha-1} \left(I^{\alpha}_{\frac{2a+b}{3}^+} f\left(\frac{a+b}{2}\right) + I^{\alpha}_{\frac{a+2b}{3}^-} f\left(\frac{a+b}{2}\right) \right), \tag{3.1} $$

where I_{σ^+} and I_{σ^-} denote the left and right Riemann-Liouville fractional integrals, respectively.

Lemma 3.1. *Assume $\alpha > 0$. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on $[a,b]$ with $a < b$. If $f' \in L^1[a,b]$, then the equality that follows is true.*

$$ \begin{aligned} &\frac{13f(a) + 27f\left(\frac{2a+b}{3}\right) + 27f\left(\frac{a+2b}{3}\right) + 13f(b)}{80} - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \mathcal{S}(f) \\ &= \frac{b-a}{9} \left(\int_0^1 \left(t^{\alpha} - \frac{39}{80}\right) f'\left((1-t)a + t\frac{2a+b}{3}\right) dt \right. \\ &\quad - \int_0^1 \frac{1}{4} (1-t)^{\alpha} f'\left((1-t)\frac{2a+b}{3} + t\frac{a+b}{2}\right) dt \\ &\quad + \int_0^1 \frac{1}{4} t^{\alpha} f'\left((1-t)\frac{a+b}{2} + t\frac{a+2b}{3}\right) dt \\ &\quad \left. - \int_0^1 \left((1-t)^{\alpha} - \frac{39}{80}\right) f'\left((1-t)\frac{a+2b}{3} + tb\right) dt \right), \end{aligned} $$

where $\mathcal{S}(f)$ is defined as in (3.1).

Proof. Let

$$ \mathcal{J} = \mathcal{J}_1 - \frac{1}{4}\mathcal{J}_2 + \frac{1}{4}\mathcal{J}_3 - \mathcal{J}_4, \tag{3.2} $$

where

$$ \mathcal{J}_1 = \int_0^1 \left(t^{\alpha} - \frac{39}{80}\right) f'\left((1-t)a + t\frac{2a+b}{3}\right) dt, $$

$$\mathcal{J}_2 = \int_0^1 (1-t)^{\alpha} f'\left((1-t)\frac{2a+b}{3} + t\frac{a+b}{2}\right) dt,$$

$$\mathcal{J}_3 = \int_0^1 t^{\alpha} f'\left((1-t)\frac{a+b}{2} + t\frac{a+2b}{3}\right) dt$$

and

$$\mathcal{J}_4 = \int_0^1 \left((1-t)^{\alpha} - \frac{39}{80}\right) f'\left((1-t)\tfrac{a+2b}{3} + tb\right) dt.$$

When we integrate by parts $\mathcal{J}_1$, we obtain

$$\begin{aligned}
\mathcal{J}_1 &= \frac{3}{b-a}\left(t^{\alpha} - \frac{39}{80}\right) f\left((1-t)a + t\frac{2a+b}{3}\right)\Big|_0^1 \\
&\quad - \frac{3\alpha}{b-a}\int_0^1 t^{\alpha-1} f\left((1-t)a + t\frac{2a+b}{3}\right) dt \\
&= \frac{123}{80(b-a)} f\left(\frac{2a+b}{3}\right) + \frac{117}{80(b-a)} f(a) - \frac{3^{\alpha+1}\alpha}{(b-a)^{\alpha+1}} \int_a^{\frac{2a+b}{3}} (u-a)^{\alpha-1} f(u)\, du \\
&= \frac{123}{80(b-a)} f\left(\frac{2a+b}{3}\right) + \frac{117}{80(b-a)} f(a) - \frac{3^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} I^{\alpha}_{\frac{2a+b}{3}^-} f(a). \qquad (3.3)
\end{aligned}$$

Likewise, we obtain

$$\mathcal{J}_2 = -\frac{6}{b-a} f\left(\frac{2a+b}{3}\right) + \frac{6^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} I^{\alpha}_{\frac{2a+b}{3}^+} f\left(\frac{a+b}{2}\right), \qquad (3.4)$$

$$\mathcal{J}_3 = \frac{6}{b-a} f\left(\frac{a+2b}{3}\right) - \frac{6^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} I^{\alpha}_{\frac{a+2b}{3}^-} f\left(\frac{a+b}{2}\right) \qquad (3.5)$$

and

$$\mathcal{J}_4 = -\frac{117}{80(b-a)} f(b) - \frac{123}{80(b-a)} f\left(\frac{a+2b}{3}\right) + \tfrac{3^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} I^{\alpha}_{\frac{a+2b}{3}^+} f(b). \qquad (3.6)$$

Using (3.3)-(3.6) in (3.2), then multiplying the resulting equality by $\frac{b-a}{9}$, we get the desired result. □

Theorem 3.2. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable function on* $[a,b]$ *with* $0 \leqslant a < b$ *such that* $f' \in L^1[a,b]$. *If* $|f'|$ *is extended s-convex for some fixed* $s \in (-1,1]$, *then for all* $\alpha > 0$ *we have*

$$\left| \frac{13f(a) + 27f\left(\frac{2a+b}{3}\right) + 27f\left(\frac{a+2b}{3}\right) + 13f(b)}{80} - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \mathcal{S}(f) \right|$$

$$\leqslant\frac{b-a}{9}\left(\left(\left(\frac{39\left(1-2\left(1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)}{80\left(s+1\right)}-B_{\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}\left(\alpha+1,s+1\right)\right.\right.\right.$$

$$\left.+B_{1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}\left(s+1,\alpha+1\right)\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)+\frac{B\left(\alpha+1,s+1\right)}{2}\left|f'\left(\frac{a+b}{2}\right)\right|$$

$$+\left(\frac{61\left(s+1\right)-39\alpha}{80\left(s+1\right)\left(\alpha+s+1\right)}+\frac{2\alpha}{\left(s+1\right)\left(\alpha+s+1\right)}\left(\frac{39}{80}\right)^{\frac{\alpha+s+1}{\alpha}}\right)$$

$$\left.\times\left(\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right)\right),$$

where $B\left(.,.\right)$ *and* $B_{\rho}\left(.,.\right)$ *are the beta and the incomplete beta functions, respectively.*

Proof. By taking the modulus of both sides of the equality given in Lemma 3.1, we have

$$\left|\frac{13f\left(a\right)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f\left(b\right)}{80}-\frac{3^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(b-a\right)^{\alpha}}\mathcal{S}\left(f\right)\right|$$

$$\leqslant\frac{b-a}{9}\left(\int_0^1\left|t^{\alpha}-\frac{39}{80}\right|\left|f'\left(\left(1-t\right)a+t\frac{2a+b}{3}\right)\right|dt\right. \tag{3.7}$$

$$+\int_0^1\frac{1}{4}\left(1-t\right)^{\alpha}\left|f'\left(\left(1-t\right)\frac{2a+b}{3}+t\frac{a+b}{2}\right)\right|dt$$

$$+\int_0^1\frac{1}{4}t^{\alpha}\left|f'\left(\left(1-t\right)\frac{a+b}{2}+t\frac{a+2b}{3}\right)\right|dt \tag{3.8}$$

$$\left.+\int_0^1\left|\left(1-t\right)^{\alpha}-\frac{39}{80}\right|\left|f'\left(\left(1-t\right)\frac{a+2b}{3}+tb\right)\right|dt\right). \tag{3.9}$$

Bu using the extended s-convexity of $|f'|$, (3.7) yields

$$\left|\frac{13f\left(a\right)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f\left(b\right)}{80}-\frac{3^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(b-a\right)^{\alpha}}\mathcal{S}\left(f\right)\right|$$

$$\leqslant\frac{b-a}{9}\left(\int_0^1\left|t^{\alpha}-\frac{39}{80}\right|\left(\left(1-t\right)^{s}\left|f'(a)\right|+t^{s}\left|f'\left(\frac{2a+b}{3}\right)\right|\right)dt\right.$$

$$+\int_0^1\frac{1}{4}\left(1-t\right)^{\alpha}\left(\left(1-t\right)^{s}\left|f'\left(\frac{2a+b}{3}\right)\right|+t^{s}\left|f'\left(\frac{a+b}{2}\right)\right|\right)dt$$

$$+\int_0^1\frac{1}{4}t^{\alpha}\left(\left(1-t\right)^{s}\left|f'\left(\frac{a+b}{2}\right)\right|+t^{s}\left|f'\left(\frac{a+2b}{3}\right)\right|\right)dt$$

$$+\int_0^1\left|(1-t)^{\alpha}-\frac{39}{80}\right|\left((1-t)^s\left|f'\left(\frac{a+2b}{3}\right)\right|+t^s\left|f'(b)\right|\right)dt\Bigg)$$

$$=\frac{b-a}{9}\Bigg(\left|f'(a)\right|\int_0^1\left|t^{\alpha}-\frac{39}{80}\right|(1-t)^s dt+\left|f'\left(\frac{2a+b}{3}\right)\right|\int_0^1\left|t^{\alpha}-\frac{39}{80}\right|t^s dt$$

$$+\left|f'\left(\frac{2a+b}{3}\right)\right|\int_0^1\frac{1}{4}(1-t)^{\alpha+s}dt+\left|f'\left(\frac{a+b}{2}\right)\right|\int_0^1\frac{1}{4}t^s(1-t)^{\alpha}dt$$

$$+\left|f'\left(\frac{a+b}{2}\right)\right|\int_0^1\frac{1}{4}t^{\alpha}(1-t)^s dt+\left|f'\left(\frac{a+2b}{3}\right)\right|\int_0^1\frac{1}{4}t^{\alpha+s}dt$$

$$+\left|f'\left(\frac{a+2b}{3}\right)\right|\int_0^1\left|(1-t)^{\alpha}-\frac{39}{80}\right|(1-t)^s dt+\left|f'(b)\right|\int_0^1\left|(1-t)^{\alpha}-\frac{39}{80}\right|t^s dt\Bigg)$$

$$=\frac{b-a}{9}\Bigg(\Bigg(\left(\frac{39}{80(s+1)}\left(1-2\left(1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)-B_{\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}(\alpha+1,s+1)\right.$$

$$\left.+B_{1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}(s+1,\alpha+1)\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)+\frac{B(\alpha+1,s+1)}{2}\left|f'\left(\frac{a+b}{2}\right)\right|$$

$$+\left(\frac{61(s+1)-39\alpha}{80(s+1)(\alpha+s+1)}+\frac{2\alpha}{(s+1)(\alpha+s+1)}\left(\frac{39}{80}\right)^{\frac{\alpha+s+1}{\alpha}}\right)$$

$$\times\left(\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right)\Bigg),$$

where we have used

$$\int_0^1\left|t^{\alpha}-\frac{39}{80}\right|(1-t)^s dt=\int_0^1\left|(1-t)^{\alpha}-\frac{39}{80}\right|t^s dt$$
$$=\frac{39\left(1-2\left(1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)}{80(s+1)}-B_{\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}(\alpha+1,s+1)$$
$$+B_{1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}(s+1,\alpha+1), \tag{3.10}$$

$$\int_0^1\left|t^{\alpha}-\frac{39}{80}\right|t^s dt=\int_0^1\left|(1-t)^{\alpha}-\frac{39}{80}\right|(1-t)^s dt$$
$$=\frac{41(s+1)-39\alpha}{80(s+1)(\alpha+s+1)}+\frac{2\alpha}{(s+1)(\alpha+s+1)}\left(\frac{39}{80}\right)^{\frac{\alpha+s+1}{\alpha}}, \tag{3.11}$$

$$\int_0^1 \frac{1}{4}(1-t)^{\alpha+s}\,dt = \int_0^1 \frac{1}{4}t^{\alpha+s}\,dt = \frac{1}{4(\alpha+s+1)} \tag{3.12}$$

and

$$\int_0^1 \frac{1}{4}t^{s}(1-t)^{\alpha}\,dt = \int_0^1 \frac{1}{4}t^{\alpha}(1-t)^{s}\,dt = \frac{1}{4}B(\alpha+1,s+1). \tag{3.13}$$

The proof is finished. □

Corollary 3.3. *In Theorem 3.2, taking* $s=0$*, we obtain*

$$\begin{aligned}&\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\mathcal{S}(f)\right|\\ &\leqslant\frac{b-a}{18(\alpha+1)}\left(\left(\left(\frac{41-39\alpha}{40}+4\alpha\left(\frac{39}{80}\right)^{1+\frac{1}{\alpha}}\right)\left(|f'(a)|+|f'(b)|\right)+\left|f'\left(\tfrac{a+b}{2}\right)\right|\right.\right.\\ &\left.+\left(\frac{61-39\alpha}{40}+4\alpha\left(\frac{39}{80}\right)^{1+\frac{1}{\alpha}}\right)\left(\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right)\right).\end{aligned}$$

Corollary 3.4. *In Theorem 3.2, taking* $s=1$*, we obtain*

$$\begin{aligned}&\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\mathcal{S}(f)\right|\\ &\leqslant\frac{b-a}{9}\left(\frac{1}{2(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|\right.\\ &+\left(\frac{160-39(\alpha+1)(\alpha+2)}{160(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{39}{80}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{\alpha+2}\left(\frac{39}{80}\right)^{1+\frac{2}{\alpha}}\right)\left(|f'(a)|+|f'(b)|\right)\\ &\left.+\left(\frac{122-39\alpha}{160(\alpha+2)}+\frac{\alpha}{(\alpha+2)}\left(\frac{39}{80}\right)^{1+\frac{2}{\alpha}}\right)\left(\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right)\right).\end{aligned}$$

Corollary 3.5. *In Theorem 3.2, taking* $\alpha=1$*, we obtain*

$$\begin{aligned}&\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\leqslant\frac{b-a}{18(s+1)(s+2)}\left(\left(\frac{39s-2}{40}+4\left(\frac{41}{80}\right)^{s+2}\right)\left(|f'(a)|+|f'(b)|\right)+\left|f'\left(\frac{a+b}{2}\right)\right|\right.\\ &\left.+\left(\frac{61s+22}{40}+4\left(\frac{39}{80}\right)^{s+2}\right)\left(\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right)\right).\end{aligned} \tag{3.14}$$

Moreover, if we choose $s=0$

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{b-a}{54}\left(\frac{1601}{1600}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)+\left|f'\left(\frac{a+b}{2}\right)\right| + \frac{2401}{1600}\left(\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right)\right). \tag{3.15}$$

Furthermore, if we choose $s=1$

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{b-a}{108}\left(\frac{187321}{128000}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)+\left|f'\left(\frac{a+b}{2}\right)\right| + \frac{324919}{128000}\left(\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right)\right). \tag{3.16}$$

Theorem 3.6. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function on* $[a,b]$ *such that* $f'\in L^1[a,b]$ *with* $0\leqslant a<b$*. If* $|f'|^q$ *is extended* s*-convex for some fixed* $s\in(-1,1]$ *and* $q>1$ *with* $\frac{1}{q}+\frac{1}{p}=1$*, then we have*

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\mathcal{S}(f)\right|$$
$$\leqslant \frac{b-a}{9}\left(\left(\left(\frac{39}{80}\right)^{p+\frac{1}{\alpha}}\frac{B\left(\frac{1}{\alpha},p+1\right)}{\alpha}+\left(\frac{41}{80}\right)^{p+1}\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{41}{80}\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}\right.$$
$$\times\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right)$$
$$\left.+\frac{1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right).$$

where $B(.,.)$ *and* ${}_2F_1(.,.,.:.)$ *are the beta and the hypergeometric functions, respectively.*

Proof. From Lemma 3.1, modulus, Hölder's inequality and the extended s-convexity of $|f'|^q$, we have

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\mathcal{S}(f)\right|$$

$$\leqslant \frac{b-a}{9}\left(\left(\int_0^1\left|t^{\alpha}-\frac{39}{80}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\,a+t\frac{2a+b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\tfrac{1}{4}\left(\int_0^1 (1-t)^{\alpha p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\,\frac{2a+b}{3}+t\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{1}{4}\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\,\frac{a+b}{2}+t\frac{a+2b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1\left|(1-t)^{\alpha}-\frac{39}{80}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\,\frac{a+2b}{3}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\right)$$

$$\leqslant \frac{b-a}{9}\left(\left(\left(\frac{39}{80}\right)^{p+\frac{1}{\alpha}}\frac{B\left(\frac{1}{\alpha},p+1\right)}{\alpha}+\left(\frac{41}{80}\right)^{p+1}\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{41}{80}\right)}{\alpha\,(p+1)}\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int_0^1\left((1-t)^s\left|f'(a)\right|^q+t^s\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\frac{1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left((1-t)^s\left|f'\left(\frac{2a+b}{3}\right)\right|^q+t^s\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\frac{1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left((1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+t^s\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\left(\left(\frac{39}{80}\right)^{p+\frac{1}{\alpha}}\frac{B\left(\frac{1}{\alpha},p+1\right)}{\alpha}+\left(\frac{41}{80}\right)^{p+1}\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{41}{80}\right)}{\alpha\,(p+1)}\right)^{\frac{1}{p}}$$

$$\left.+\left(\int_0^1\left((1-t)^s\left|f'\left(\frac{a+2b}{3}\right)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right)$$

$$=\frac{b-a}{9}\left(\left(\left(\frac{39}{80}\right)^{p+\frac{1}{\alpha}}\frac{B\left(\frac{1}{\alpha},p+1\right)}{\alpha}+\left(\frac{41}{80}\right)^{p+1}\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{41}{80}\right)}{\alpha\,(p+1)}\right)^{\frac{1}{p}}\right.$$

$$\times\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right)$$

$$+\frac{1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\Bigg),$$

where we have used that

$$\begin{aligned}\int_0^1|t^\alpha-\varepsilon|^p\,dt=&\frac{\varepsilon^{p+\frac{1}{\alpha}}}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}(1-t)^p\,dt+\frac{(1-\varepsilon)^{p+1}}{\alpha}\int_0^1(1-t)^p\left(1-(1-\varepsilon)t\right)^{\frac{1}{\alpha}-1}dt\\=&\varepsilon^{p+\frac{1}{\alpha}}\frac{B\left(\frac{1}{\alpha},p+1\right)}{\alpha}+(1-\varepsilon)^{p+1}\frac{{}_2F_1\left(1-\frac{1}{\alpha},1,p+2,1-\varepsilon\right)}{\alpha(p+1)}.\end{aligned}$$

The proof is finished. □

Corollary 3.7. *In Theorem 3.6, taking $s=0$, we obtain*

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\mathcal{S}(f)\right|$$

$$\leqslant\frac{b-a}{9}\left(\left(\left(\frac{39}{80}\right)^{p+\frac{1}{\alpha}}\frac{B\left(\frac{1}{\alpha},p+1\right)}{\alpha}+\left(\frac{41}{80}\right)^{p+1}\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{41}{80}\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}\right.$$

$$\times\left(\left(|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right)$$

$$+\frac{1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)^{\frac{1}{q}}\right)\right).$$

Corollary 3.8. *In Theorem 3.6, taking $s=1$, we obtain*

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\mathcal{S}(f)\right|$$

$$\leqslant\frac{b-a}{9}\left(\left(\left(\frac{39}{80}\right)^{p+\frac{1}{\alpha}}\frac{B\left(\frac{1}{\alpha},p+1\right)}{\alpha}+\left(\frac{41}{80}\right)^{p+1}\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{41}{80}\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}\right.$$

$$\times\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right)$$

$$+\frac{1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right)\Bigg).$$

Corollary 3.9. *In Theorem 3.6, taking $\alpha = 1$, we obtain*

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leqslant\frac{b-a}{9}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\left(\frac{39}{80}\right)^{p+1}+\left(\frac{41}{80}\right)^{p+1}\right)^{\frac{1}{p}}\right.$$

$$\times\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right)$$

$$+\frac{1}{4}\left(\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\Bigg).$$

Moreover, if we choose $s = 0$

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leqslant\frac{b-a}{9}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{39^{p+1}+41^{p+1}}{80}\right)^{\frac{1}{p}}\right.$$

$$\times\left(\frac{\left(|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}}{80}\right)$$

$$+\left.\frac{\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)^{\frac{1}{q}}}{4}\right).$$

Furthermore, if we choose $s = 1$

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leqslant \frac{b-a}{9}\left(\tfrac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{39^{p+1}+41^{p+1}}{80^{p+1}}\right)^{\frac{1}{p}}\right.$$
$$\times\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right)$$
$$\left.+\frac{1}{4}\left(\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right)\right).$$

Theorem 3.10. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function on* $[a,b]$ *such that* $f'\in L^1[a,b]$ *with* $0\leqslant a<b$. *If* $|f'|^q$ *is extended s-convex in the second sense for some fixed* $s\in(0,1]$ *and* $q\geqslant 1$, *then we have*

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\mathcal{S}(f)\right|$$
$$\leqslant\frac{b-a}{9}\left(\left(\frac{41-39\alpha}{80(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{39}{80}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\left(\left(\left(\frac{39\left(1-2\left(1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)}{80(s+1)}-B_{\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}(\alpha+1,s+1)+B_{1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}(s+1,\alpha+1)\right)|f'(a)|^q\right.\right.\right.$$
$$\left.+\left(\frac{41(s+1)-39\alpha}{80(s+1)(\alpha+s+1)}+\frac{2\alpha}{(s+1)(\alpha+s+1)}\left(\frac{39}{80}\right)^{\frac{\alpha+s+1}{\alpha}}\right)\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\left(\frac{41(s+1)-39\alpha}{80(s+1)(\alpha+s+1)}+\frac{2\alpha}{(s+1)(\alpha+s+1)}\left(\frac{39}{80}\right)^{\frac{\alpha+s+1}{\alpha}}\right)\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right.$$
$$\left.\left.+\left(\frac{39\left(1-2\left(1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)}{80(s+1)}-B_{\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}(\alpha+1,s+1)+B_{1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}(s+1,\alpha+1)\right)|f'(b)|^q\right)^{\frac{1}{q}}\right)$$
$$+\frac{1}{4}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\left(\frac{1}{\alpha+s+1}\left|f'\left(\frac{2a+b}{3}\right)\right|^q+B(s+1,\alpha+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(B(s+1,\alpha+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\tfrac{1}{\alpha+s+1}\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)^{\frac{1}{q}}\right)\right)$$

where $B(.,.)$ *and* $B_{\rho}(.,.)$ *are the beta and the incomplete beta functions, respectively.*

Proof. From Lemma 3.1, modulus, power mean inequality, the extended s-convexity of $|f'|^q$ and using (2.8)-(2.11), we get

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\mathcal{S}(f)\right|$$

$$\leqslant\frac{b-a}{9}\left(\left(\int_0^1\left|t^{\alpha}-\frac{39}{80}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t^{\alpha}-\frac{39}{80}\right|\left|f'\left((1-t)a+t\frac{2a+b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\frac{1}{4}\left(\int_0^1(1-t)^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{\alpha}\left|f'\left((1-t)\frac{2a+b}{3}+t\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{1}{4}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left|f'\left((1-t)\frac{a+b}{2}+t\frac{a+2b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1\left|(1-t)^{\alpha}-\frac{39}{80}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|(1-t)^{\alpha}-\frac{39}{80}\right|\left|f'\left((1-t)\frac{a+2b}{3}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\right)$$

$$\leqslant\frac{b-a}{9}\left(\left(\int_0^1\left|t^{\alpha}-\frac{39}{80}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|t^{\alpha}-\frac{39}{80}\right|\left((1-t)^s|f'(a)|^q+t^s\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$+\frac{1}{4}\left(\int_0^1(1-t)^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{\alpha}\left((1-t)^s\left|f'\left(\frac{2a+b}{3}\right)\right|^q+t^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\frac{1}{4}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left((1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+t^s\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1\left|(1-t)^{\alpha}-\frac{39}{80}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|(1-t)^{\alpha}-\frac{39}{80}\right|\left((1-t)^s\left|f'\left(\frac{a+2b}{3}\right)\right|^q+t^s|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right)$$

$$=\frac{b-a}{9}\left(\left(\frac{41-39\alpha}{80(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{39}{80}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\left(\left(\left(\frac{39\left(1-2\left(1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)}{80(s+1)}-B_{\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}(\alpha+1,s+1)+B_{1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}(s+1,\alpha+1)\right)\left|f'(a)\right|^{q}\right.\right.\right.$$

$$\left.+\left(\frac{41(s+1)-39\alpha}{80(s+1)(\alpha+s+1)}+\frac{2\alpha}{(s+1)(\alpha+s+1)}\left(\frac{39}{80}\right)^{\frac{\alpha+s+1}{\alpha}}\right)\left|f'\left(\frac{2a+b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}$$

$$+\left(\left(\frac{41(s+1)-39\alpha}{80(s+1)(\alpha+s+1)}+\frac{2\alpha}{(s+1)(\alpha+s+1)}\left(\frac{39}{80}\right)^{\frac{\alpha+s+1}{\alpha}}\right)\left|f'\left(\frac{a+2b}{3}\right)\right|^{q}\right.$$

$$\left.\left.+\left(\frac{39\left(1-2\left(1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)}{80(s+1)}-B_{\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}(\alpha+1,s+1)+B_{1-\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}}(s+1,\alpha+1)\right)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right)$$

$$+\frac{1}{4}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\left(\frac{1}{\alpha+s+1}\left|f'\left(\frac{2a+b}{3}\right)\right|^{q}+B(s+1,\alpha+1)\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(B(s+1,\alpha+1)\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+\frac{1}{\alpha+s+1}\left|f'\left(\frac{a+2b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right)\right).$$

where we have used (3.10)-(3.13). The proof is completed. □

Corollary 3.11. *In Theorem 3.10, taking $s=0$, we obtain*

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\mathcal{S}(f)\right|$$

$$\leqslant\frac{b-a}{9(\alpha+1)}\left(\left(41-39\alpha\left(1-2\left(\frac{39}{80}\right)^{\frac{1}{\alpha}}\right)\right)\right.$$

$$\times\left(\frac{\left(|f'(a)|^{q}+\left|f'\left(\frac{2a+b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^{q}+|f'(b)|^{q}\right)^{\frac{1}{q}}}{80}\right.$$

$$\left.+\frac{\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^{q}+\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+\left|f'\left(\frac{a+2b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}}{4}\right).$$

Corollary 3.12. *In Theorem 3.10, taking $s=1$, we obtain*

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\mathcal{S}(f)\right|$$

$$\leqslant \frac{b-a}{9}\left(\left(\frac{41-39\alpha}{80(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{39}{80}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\left(\left(\left(\frac{160-39(\alpha+1)(\alpha+2)}{160(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{39}{80}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{\alpha+2}\left(\frac{39}{80}\right)^{1+\frac{2}{\alpha}}\right)\left|f'(a)\right|^q\right.\right.\right.$$

$$\left.+\left(\frac{82-39\alpha}{160(\alpha+2)}+\frac{\alpha}{(\alpha+2)}\left(\frac{39}{80}\right)^{1+\frac{2}{\alpha}}\right)\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\left(\frac{82-39\alpha}{160(\alpha+2)}+\frac{\alpha}{(\alpha+2)}\left(\frac{39}{80}\right)^{1+\frac{2}{\alpha}}\right)\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right.$$

$$\left.\left.+\left(\frac{160-39(\alpha+1)(\alpha+2)}{160(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{39}{80}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{\alpha+2}\left(\frac{39}{80}\right)^{1+\frac{2}{\alpha}}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right)$$

$$+\frac{1}{4(\alpha+1)}\left(\left(\frac{\alpha+1}{\alpha+2}\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\frac{1}{\alpha+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\frac{1}{\alpha+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{\alpha+1}{\alpha+2}\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)^{\frac{1}{q}}\right)\right).$$

Corollary 3.13. *In Theorem 3.10, taking $\alpha=1$, we obtain*

$$\left|\frac{13f(a)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f(b)}{80}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\leqslant\frac{b-a}{9}\left(\left(\frac{1601}{6400}\right)^{1-\frac{1}{q}}\left(\left(\left(\frac{39s-2}{80(s+1)(s+2)}+\frac{2}{(s+1)(s+2)}\left(\frac{41}{80}\right)^{s+2}\right)\left|f'(a)\right|^q\right.\right.\right.$$

$$\left.+\left(\frac{41s+2}{80(s+1)(s+2)}+\frac{2}{(s+1)(s+2)}\left(\frac{39}{80}\right)^{s+2}\right)\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\left(\frac{41s+2}{80(s+1)(s+2)}+\frac{2}{(s+1)(s+2)}\left(\frac{39}{80}\right)^{s+2}\right)\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right.$$

$$\left.\left.+\left(\frac{39s-2}{80(s+1)(s+2)}+\frac{2}{(s+1)(s+2)}\left(\frac{41}{80}\right)^{s+2}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right)$$

$$+\frac{1}{8}\left(\left(\frac{2(s+1)\left|f'\left(\frac{2a+b}{3}\right)\right|^q+2\left|f'\left(\frac{a+b}{2}\right)\right|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{2\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+2\left(s+1\right)\left|f'\left(\frac{a+2b}{3}\right)\right|^{q}}{\left(s+1\right)\left(s+2\right)}\right)^{\frac{1}{q}}\Bigg)\Bigg).$$

Moreover, if we choose $s=0$

$$\left|\frac{13f\left(a\right)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f\left(b\right)}{80}-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right|$$

$$\leqslant\frac{b-a}{9}\left(\frac{1601\left(\left(\left|f'\left(a\right)\right|^{q}+\left|f'\left(\frac{2a+b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^{q}+\left|f'\left(b\right)\right|^{q}\right)^{\frac{1}{q}}\right)}{6400}\right.$$

$$\left.+\frac{\left(\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^{q}+\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+\left|f'\left(\frac{a+2b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right)}{8}\right).$$

Furthermore, if we choose $s=1$

$$\left|\frac{13f\left(a\right)+27f\left(\frac{2a+b}{3}\right)+27f\left(\frac{a+2b}{3}\right)+13f\left(b\right)}{80}-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right|$$

$$\leqslant\frac{1601\left(b-a\right)}{57600}\Bigg(\Bigg(\left(\frac{187321\left|f'\left(a\right)\right|^{q}+196919\left|f'\left(\frac{2a+b}{3}\right)\right|^{q}}{384240}\right)^{\frac{1}{q}}$$

$$+\left(\frac{196919\left|f'\left(\frac{2a+b}{3}\right)\right|^{q}+187321\left|f'\left(b\right)\right|^{q}}{384240}\right)^{\frac{1}{q}}\Bigg)$$

$$+\frac{800}{1601}\left(\left(\frac{2\left|f'\left(\frac{2a+b}{3}\right)\right|^{q}+\left|f'\left(\frac{a+b}{2}\right)\right|^{q}}{3}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+2\left|f'\left(\frac{a+2b}{3}\right)\right|^{q}}{3}\right)^{\frac{1}{q}}\right)\Bigg).$$

4. Applications to special means

We shall consider the means for arbitrary real numbers a,b,c

The arithmetic mean: $A\left(a,b\right)=\frac{a+b}{2}, A\left(a,b,c\right)=\frac{a+b+c}{3}$.

The harmonic mean: $H\left(a,b\right)=\frac{2}{\frac{1}{a}+\frac{1}{b}}=\frac{2ab}{a+b}$, $H\left(a,b,c\right)=\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\frac{3abc}{ac+bc+ab}$, $a,b,c>0$.

The geometric mean: $G\left(a,b\right)=\sqrt{ab}$, $a,b>0$

The logarithmic means: $L\left(a,b\right)=\frac{b-a}{\ln b-\ln a}$, $a,b>0$ and $a\neq b$.

The p-Logarithmic mean: $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a,b > 0, a \neq b$ and $p \in \mathbb{R}\backslash\{-1,0\}$.

Proposition 4.1. *Let* $a,b,s \in \mathbb{R}$ *with* $0 < a < b$ *and* $s \in (0,1)$, *then we have*

$$\begin{aligned}
&\left|13A\left(a^{s+1},b^{s+1}\right)+27A\left(A^{s+1}(a,a,b),A^{s+1}(a,b,b)\right)-40L_{s+1}^{s+1}(a,b)\right| \\
\leqslant &\frac{b-a}{18(s+2)}\left(\frac{\left(39\times 6^{s}+61\left(2^{s}+4^{s}\right)\right)s+22\left(2^{s}+4^{s}\right)+40\times 3^{s}-2\times 6^{s}}{6^{s}}\right. \\
&\left.+\frac{41^{s+2}+39^{s+2}\left(1+2^{s}\right)}{40^{s+1}\times 6^{s}}\right)\left(a^{s}+b^{s}\right).
\end{aligned}$$

Proof. From inequality (3.15) with $f(t) = \frac{1}{s+1}t^{s+1}$ which the derivative is s-convex, we have

$$\begin{aligned}
&\left|13A\left(a^{s+1},b^{s+1}\right)+27A\left(A^{s+1}(a,a,b),A^{s+1}(a,b,b)\right)-40L_{s+1}^{s+1}(a,b)\right| \\
\leqslant &\frac{20(b-a)}{9(s+2)}\left(\left(\frac{39s-2}{40}+4\left(\frac{41}{80}\right)^{s+2}\right)\left(a^{s}+b^{s}\right)+\left(\frac{a+b}{2}\right)^{s}\right. \\
&\left.+\left(\frac{61s+22}{40}+4\left(\frac{39}{80}\right)^{s+2}\right)\left(\left(\frac{2a+b}{3}\right)^{s}+\left(\frac{a+2b}{3}\right)^{s}\right)\right).
\end{aligned}$$

Combining the above inequality with $(a+b)^s \leqslant (a^s+b^s)$ since $s<1$, we get the desired result. □

Proposition 4.2. *Let* $a,b \in \mathbb{R}$ *with* $0 < a < b$, *then we have*

$$\begin{aligned}
&\left|26H^{-1}(a,b)+27G^{-2}(a,b)\left(H(a,b,b)+H(a,a,b)\right)-80L^{-1}(a,b)\right| \\
\leqslant &\frac{b-a}{54}\left(\frac{1601}{20}\left(\frac{1}{a^{2}}+\frac{1}{b^{2}}\right)+\frac{4}{(a+b)^{2}}+\frac{2401}{20}\left(\frac{9}{(2a+b)^{2}}+\frac{9}{(a+2b)^{2}}\right)\right).
\end{aligned}$$

Proof. The assertion follows from inequality (3.16), applied to the function $f(t) = \frac{1}{t}$, which derivative is P-function. □

Proposition 4.3. *Let* $a,b,n \in \mathbb{R}$ *with* $0 < a < b$ *and* $n \geqslant 2$, *then we have*

$$\begin{aligned}
&\left|26A\left(a^{n},b^{n}\right)+27A^{n}(a,a,b)+27A^{n}(a,b,b)-80L_{n}^{n}(a,b)\right| \\
\leqslant &\frac{(b-a)n}{108}\left(\frac{187321}{1600}\left(a^{n-1})+b^{n-1}\right)\right)+80\left(\frac{a+b}{2}\right)^{n-1} \\
&\left.+\frac{324919}{1600}\left(\left(\frac{2a+b}{3}\right)^{n-1}+\left(\frac{a+2b}{3}\right)^{n-1}\right)\right). \qquad (4.1)
\end{aligned}$$

Proof. The assertion follows from inequality (3.16), applied to the function $f(t) = t^n$ which derivative is convex. □

5. Conclusion

In conclusion, this work contributes to the growing body of research on fractional integral inequalities by introducing refined versions of the corrected Simpson's second formula under the extended s-convexity assumption. The key identity involving Riemann–Liouville fractional integrals not only enables the derivation of new estimates but also provides a versatile tool for future investigations. The applications to special means illustrate the concrete analytical value of our results, particularly in contexts where convexity-based error bounds are essential. We believe these findings open avenues for further generalizations, including extensions to other types of fractional operators (e.g., Caputo, Hadamard) or more general convexity notions such as multiplicative or quantum convexity.

Conflict of interest The authors declare that they do not have any conflicts of interest.

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