Journal of Fractional Calculus and Nonlinear Systems
jfcns.sabapub.com
ISSN : 2709-9547

J Frac Calc & Nonlinear Sys
(2025)6(2) : 16-64
doi:10.48185/jfcns.v6i2.1726

Stability and Simulation of Fractional Dynamics in Nonhomogeneous Media via Generalized Quantum-Hadamard Operators

Rabha W. Ibrahim[α*]
[α]Information and Communication Technology Research Group, Scientific Research Center, Al-Ayen University, Iraq

• *Received: 29 July 2025* • *Accepted: 29 November 2025* • *Published Online: 28 December 2025*

Abstract

In this study, we extend the concept of the quantum Gamma function (q-Gamma) by introducing a new (q, τ)-deformed Gamma function. This generalization allows us to construct an enriched family of Hadamard-type fractional operators, which we then apply to the analysis of memory and decoherence in open quantum systems. The inclusion of the deformation parameter q together with the delay-like scaling parameter τ makes the proposed (q, τ)-Hadamard framework particularly suited to capture nonlocal and non-Markovian effects, thereby offering a flexible tool for describing structured reservoirs and anomalous dissipation. To investigate the analytical consequences, we employ a (q, τ)-Mittag–Leffler function, through which explicit solutions are obtained for a class of fractional differential equations, with particular emphasis on population dynamics models. These solutions reveal a variety of memory-driven features, including sub-exponential decay of coherence and the occurrence of revival phenomena. Both the fractional order α and the deformation parameters play a decisive role in shaping the temporal behavior. Beyond population models, the framework also provides insights into fractional quantum master equations, quantum walks with fractional memory, and noise effects in quantum information processing.

Keywords: Fractional calculus; Quantum memory; Non-Markovian dynamics; Decoherence.

1. Introduction

By eliminating the need for limits and infinitesimals, quantum calculus also known as *calculus without limits* or q-*calculus* generalizes classical calculus. It introduces the q-*derivative*, defined as follows ([1, 2]):

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad q \neq 1, \quad x \neq 0.$$

As $q \to 1$, the q-derivative converges to the classical derivative:

$$\lim_{q \to 1} D_q f(x) = f'(x).$$

*Corresponding author: rabhaibrahim@yahoo.com

Quantum calculus finds applications in number theory, quantum groups, combinatorics, and special functions (such as q-hypergeometric functions). It also provides discrete analogs of classical calculus, which are useful in approximation theory and quantum mechanics [3, 4]. Hadamard calculus, a variant of fractional calculus, defines differentiation and integration in terms of the logarithm of the independent variable. According to [5, 6], the *Hadamard fractional derivative* of order $\alpha > 0$ is given by:

$$\left({}^{H}D^{\alpha}f\right)(x) = \frac{1}{\Gamma(n-\alpha)}\, x\frac{d^n}{dx^n}\int_1^x \left(\log\frac{x}{t}\right)^{n-\alpha-1}\frac{f(t)}{t}\,dt,$$

where $n = \lceil\alpha\rceil$ is the smallest integer greater than or equal to α. Unlike standard fractional derivatives such as the Riemann–Liouville or Caputo types, the Hadamard derivative emphasizes scale invariance and is naturally suited to functions defined on the interval $(0, \infty)$ [6, 7, 8]. Hadamard fractional calculus has been widely employed to model physical and engineering processes with scaling properties, as well as to solve differential equations with variable coefficients [9, 10]. A comparison of classical, quantum, and Hadamard calculus is summarized in Table 1. Understanding decoherence, dissipation,

Table 1: An overview of the structure of (q, τ)-Hadamard operators and their relationship to fractional quantum theory and quantum computation.

Aspect	**Quantum Calculus**	**Hadamard Calculus**
Derivative Type	q-difference operator	Fractional derivative w.r.t. $\log x$
Limit Process	Avoids classical limits	Uses integrals involving logarithms
Applications	Quantum groups, special functions, combinatorics	FDEs, scale-invariant processes
Key Feature	Discrete-like calculus with parameter q	Logarithmic scale fractional calculus

and quantum control now heavily relies on the study of open quantum systems, or quantum systems that interact with their surroundings. Because of entanglement with ambient degrees of freedom, open systems usually display non-unitary dynamics in contrast to closed systems that are subject to unitary evolution. This results in processes like energy dissipation and decoherence, which are important in condensed matter physics, quantum information processing, and quantum optics [11, 12]. By ignoring memory effects and presuming that the system's future evolution is only dependent on its current state, classical techniques frequently make the assumption that the setting is Markovian. Nonetheless, a large number of genuine quantum systems, including solid-state qubits, spin chains, and photonic crystals, interact with correlated or structured surroundings that display long-range temporal correlations. To capture memory effects in their dynamical evolution, these non-Markovian systems need more advanced methods [13, 14].

A strong mathematical foundation for describing memory and anomalous transport in a variety of domains, including as diffusion processes, control theory, and viscoelasticity, is fractional calculus, especially when combined with non-integer derivatives and integrals. Sub-exponential decay of coherence and anomalous relaxation have been satisfactorily described using fractional differential equations, which naturally encode non-locality in time in the context of quantum systems [15, 16].

The traditional Hadamard fractional operator is extended to a novel (q, τ)-deformed situation in this study. The (q, τ)-Gamma function is introduced, as follows (see [17, 18, 19, 20, 21, 22, 23]):

$$\Gamma_{q,\tau}(\alpha) = (1-q)^{1-\alpha} \prod_{n=0}^{\infty} \frac{1-q^{\tau(n+1)}}{1-q^{\tau(\alpha+n)}}.$$

To unify scale-invariant memory effects with deformation and delay structures, we introduce generalized (q,τ)-Hadamard-type fractional integrals and derivatives. These operators provide a more flexible framework for modeling quantum systems interacting with complex or non-trivial environments. In particular, we apply this generalized calculus to investigate the decoherence dynamics of a two-level system embedded in a non-Markovian setting. The system's evolution is described by a fractional differential equation involving the (q,τ)-Hadamard derivative, with solutions expressed in terms of a novel (q,τ)-Mittag-Leffler function. The resulting dynamics exhibit memory-dependent features, including reduced decoherence rates and partial coherence revivals characteristics not captured by standard Markovian models. Furthermore, the proposed approach has potential applications in modeling quantum information transmission through structured or fractal media, quantum master equations with memory kernels, and fractional quantum walks. This study lays the groundwork for employing fractional and deformed operator techniques in the design of memory-resilient quantum systems and algorithms.

2. Generalized quantum fractional calculus

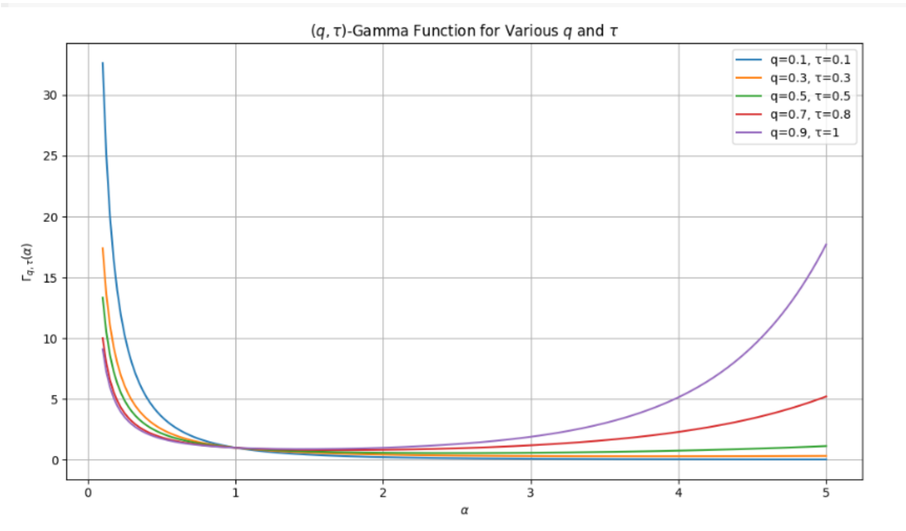

Figure 1: The graph of $\Gamma_{q,\tau}(\alpha)$.

Definition 2.1. Let $0 < q < 1$ and $\tau > 0$. The (q,τ)-Gamma function is defined as (see Fig.1):

$$\Gamma_{q,\tau}(\alpha) = (1-q)^{1-\alpha} \prod_{n=0}^{\infty} \frac{1-q^{\tau(n+1)}}{1-q^{\tau(\alpha+n)}}, \quad \Re(\alpha) > 0.$$

The classical Euler Gamma function is generalized by this function, which satisfies $\tau = 1$ $\lim_{q\to 1^-} \Gamma_{q,\tau}(\alpha) = \Gamma(\alpha)$.

Proposition 2.2 (Properties of $\Gamma_{q,\tau}(\alpha)$)**.**

1. ***Classical Limit:***
$$\lim_{q\to 1^-} \Gamma_{q,\tau}(\alpha) = \Gamma(\alpha), \quad \textit{for } \tau = 1.$$
2. ***Normalization:***
$$\Gamma_{q,\tau}(1) = 1.$$
3. ***Recurrence Relation:***
$$\Gamma_{q,\tau}(\alpha+1) = \frac{1-q^{\tau\alpha}}{(1-q)^{\tau}}\,\Gamma_{q,\tau}(\alpha).$$

Proof. We aim to derive the recurrence relation. Consider the following definition
$$\Gamma_{q,\tau}(\alpha+1) = (1-q)^{1-(\alpha+1)} \prod_{n=0}^{\infty} \frac{1-q^{\tau(n+1)}}{1-q^{\tau(\alpha+1+n)}}.$$
Observe that
$$(1-q)^{1-(\alpha+1)} = (1-q)^{-\alpha} = (1-q)^{\tau}\cdot(1-q)^{1-\alpha-\tau}.$$
Next, note that the denominator in the product can be shifted:
$$\prod_{n=0}^{\infty} \frac{1-q^{\tau(n+1)}}{1-q^{\tau(\alpha+1+n)}} = \frac{1-q^{\tau\alpha}}{1-q^{\tau}} \cdot \prod_{n=1}^{\infty} \frac{1-q^{\tau(n+1)}}{1-q^{\tau(\alpha+n)}},$$
since the $n = 0$ term in the denominator becomes $1-q^{\tau(\alpha+1)}$, and comparing with the original $\Gamma_{q,\tau}(\alpha)$:
$$\Gamma_{q,\tau}(\alpha) = (1-q)^{1-\alpha} \prod_{n=0}^{\infty} \frac{1-q^{\tau(n+1)}}{1-q^{\tau(\alpha+n)}}.$$
Thus, we obtain:
$$\Gamma_{q,\tau}(\alpha+1) = \frac{1-q^{\tau\alpha}}{(1-q)^{\tau}} \cdot \Gamma_{q,\tau}(\alpha),$$
which completes the derivation. □

Moreover, we have the following result:

Proposition 2.3.

1. ***Special Values (Integer Argument):*** *For* $n \in \mathbb{N}$*,*
$$\Gamma_{q,\tau}(n) = \frac{(1-q)^{1-n}}{(q^{\tau};q^{\tau})_{n-1}},$$
where $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ *denotes the* q*-Pochhammer symbol.*
2. ***Logarithmic Derivative (Digamma Function):*** *Define the* (q,τ)*-digamma function:*
$$\psi_{q,\tau}(\alpha) := \frac{d}{d\alpha}\log\Gamma_{q,\tau}(\alpha).$$
Then,
$$\psi_{q,\tau}(\alpha) = -\tau\log(1-q) + \tau\sum_{n=0}^{\infty} \frac{q^{\tau(\alpha+n)}\log q}{1-q^{\tau(\alpha+n)}}.$$

3. ***Multiplicative Scaling:*** *For scalar* $c > 0$*, the Gamma function satisfies a scaled argument relation:*
$$\Gamma_{q,\tau}(c\alpha) \neq c^{\alpha-1}\Gamma_{q,\tau}(\alpha).$$
4. ***Log-Convexity:*** *The function* $\Gamma_{q,\tau}(\alpha)$ *is log-convex for* $\alpha > 0$*, i.e.,*
$$\frac{d^2}{d\alpha^2}\log\Gamma_{q,\tau}(\alpha) > 0.$$

The classical and fundamental (Jackson-type) q-Gamma functions are generalized by the function $\Gamma_{q,\tau}(\alpha)$. In analytic function theory and quantum calculus, it is helpful for defining new convolution integrals, special functions, and deformed fractional operators.

2.1. Q-Fractional Hadamard opeartors

Definition 2.4. The (q,τ)-Gamma function is used to define generalized Hadamard-type fractional integral and differential operators, which we explore. Let f be a function defined on $(0,\infty)$, and let $\alpha > 0$. The following is the definition of the (q,τ)-Hadamard fractional integral of order α:

$$\left({}^{H}I^{\alpha}_{q,\tau}f\right)(x) = \frac{1}{\Gamma_{q,\tau}(\alpha)}\int_1^x \left(\log\frac{x}{t}\right)^{\alpha-1}\frac{f(t)}{t}\,dt, \quad x > 1.$$

For order $\alpha > 0$, the equivalent (q,τ)-Hadamard fractional derivative is provided by:

$$\left({}^{H}D^{\alpha}_{q,\tau}f\right)(x) = x\frac{d}{dx}\left(I^{1-\alpha}_{H;q,\tau}f\right)(x) = \frac{x}{\Gamma_{q,\tau}(1-\alpha)}\frac{d}{dx}\int_1^x \left(\log\frac{x}{t}\right)^{-\alpha}\frac{f(t)}{t}\,dt.$$

Remark 2.5. The operators decrease to the standard Hadamard fractional integral and derivative when $q \to 1^-$ and $\tau = 1$. In the (q,τ)-deformed calculus, these specialized operators offer a natural framework for researching scale-invariant or logarithmic memory processes. They can be applied to define convolution-type integral transforms and new classes of differential equations.

Remark 2.6. Semigroup Property can be realized as follows: let $\alpha, \beta > 0$. Then under suitable regularity conditions on f,

$${}^{H}I^{\alpha}_{q,\tau}\left[{}^{H}I^{\beta}_{q,\tau}f(x)\right] = {}^{H}I^{\alpha+\beta}_{q,\tau}f(x).$$

This is inferred from the multiplicative property of the (q,τ)-Gamma function and the associativity of the logarithmic convolution kernel:

$$\Gamma_{q,\tau}(\alpha)\Gamma_{q,\tau}(\beta) \sim \Gamma_{q,\tau}(\alpha+\beta)\cdot(\text{normalization}).$$

One way to think of the Inversion Formula is to use $D^{\alpha}_{H;q,\tau}$ to represent the (q,τ)-Hadamard fractional derivative. Then, in the right circumstances,

$${}^{H}D^{\alpha}_{q,\tau}\left[{}^{H}I^{\alpha}_{q,\tau}f(x)\right] = f(x), \quad {}^{H}I^{\alpha}_{q,\tau}\left[{}^{H}D^{\alpha}_{q,\tau}f(x)\right] = f(x).$$

The (q,τ)-Laplace-Type transform can be formulated by the fact:

$$\mathcal{L}_{q,\tau}\{f(x)\}(s) = \int_1^{\infty} f(x)x^{-s}\,\frac{dx}{x}.$$

Then,

$$\mathcal{L}_{q,\tau}\{{}^{H}I^{\alpha}_{q,\tau}f(x)\}(s) = \frac{\Gamma_{q,\tau}(\alpha)}{s^{\alpha}}\mathcal{L}_{q,\tau}\{f(x)\}(s).$$

Define the generalized Mittag-Leffler function:

$$E^{(q,\tau)}_{\alpha}(x) = \sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma_{q,\tau}(\alpha n+1)}.$$

This function solves fractional equations of the form:

$${}^{H}D^{\alpha}_{q,\tau}y(x) = \lambda y(x).$$

3. Solutions of (q,τ)-Hadamard Fractional Differential Equations

Take the linear (q,τ)-Hadamard fractional differential equation of order $\alpha > 0$ as an example:

$${}^{H}D^{\alpha}_{q,\tau}y(x) = \lambda y(x) + f(x), \quad x > 1,$$

via the initial condition

$$y(1) = y_0,$$

where ${}^{H}D^{\alpha}_{q,\tau}$ is the(q,τ)-Hadamard fractional derivative, $\lambda \in \mathbb{C}$ is a constant, $f(x)$ is a given forcing function. We proceed to determine the solution by using (q,τ)-Laplace-type transform. Define the transform

$$\mathcal{L}_{q,\tau}\{y(x)\}(s) = \int_{1}^{\infty} y(x)\, x^{-s}\,\frac{dx}{x}.$$

Applying $\mathcal{L}_{q,\tau}$ to both sides of the equation, we have

$$\mathcal{L}_{q,\tau}\{{}^{H}D^{\alpha}_{q,\tau}y(x)\}(s) = \lambda\mathcal{L}_{q,\tau}\{y(x)\}(s) + \mathcal{L}_{q,\tau}\{f(x)\}(s).$$

Utilizing the transform property of the fractional derivative,

$$s^{\alpha}\mathcal{L}_{q,\tau}\{y(x)\}(s) - s^{\alpha-1}y(1) = \lambda\mathcal{L}_{q,\tau}\{y(x)\}(s) + \mathcal{L}_{q,\tau}\{f(x)\}(s).$$

Rearranging for $\mathcal{L}_{q,\tau}\{y(x)\}(s)$,

$$\mathcal{L}_{q,\tau}\{y(x)\}(s) = \frac{s^{\alpha-1}y_0 + \mathcal{L}_{q,\tau}\{f(x)\}(s)}{s^{\alpha}-\lambda}.$$

Next, we aim to get the solution in terms of the generalized Mittag-Leffler Function. For the homogeneous case $f(x) \equiv 0$, the inverse transform gives

$$y(x) = y_0\, E^{(q,\tau)}_{\alpha}\big(\lambda(\log x)^{\alpha}\big).$$

For the nonhomogeneous case, the solution can be expressed as

$$y(x) = y_0\, E^{(q,\tau)}_{\alpha}\big(\lambda(\log x)^{\alpha}\big) + \int_{1}^{x}\left(\log\frac{x}{t}\right)^{\alpha-1}\frac{1}{\Gamma_{q,\tau}(\alpha)}f(t)\,\frac{dt}{t},$$

where the integral term owns the (q,τ)-Hadamard fractional integral of the forcing function f. The Existence and Uniqueness Theorem is formulated in the next result.

Theorem 3.1. *Suppose* $f : [1, \infty) \to \mathbb{R}$ *is continuous and satisfies a growth condition such that the integral*

$$ {}^{H}I^{\alpha}_{q,\tau} f(x) = \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_1^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t}\, dt $$

is well defined and continuous for all $x \geqslant 1$. *Then the initial value problem*

$$ {}^{H}D^{\alpha}_{q,\tau} y(x) = \lambda y(x) + f(x), \quad y(1) = y_0, $$

has a unique continuous solution on $[1, \infty)$ *given by*

$$ y(x) = y_0 E^{(q,\tau)}_{\alpha}\big(\lambda (\log x)^{\alpha}\big) + ^{H} I^{\alpha}_{q,\tau} f(x). $$

Proof.

Step 1: Reformulate the problem as an integral equation.

Recall that the (q, τ)-Hadamard fractional derivative and integral operators satisfy the inversion formula

$$ {}^{H}D^{\alpha}_{q,\tau} \left({}^{H}I^{\alpha}_{q,\tau} g(x)\right) = g(x) $$

for suitable functions g. Apply the fractional integral ${}^{H}I^{\alpha}_{q,\tau}$ of order α to both sides of the differential equation:

$$ {}^{H}I^{\alpha}_{q,\tau} \left({}^{H}D^{\alpha}_{q,\tau} y(x)\right) = ^{H} I^{\alpha}_{q,\tau}\big(\lambda y(x) + f(x)\big). $$

Utilizing the inversion property on the left side,

$$ y(x) = y(1) + \lambda {}^{H}I^{\alpha}_{q,\tau} y(x) + ^{H} I^{\alpha}_{q,\tau} f(x). $$

Note that by the definition of fractional integral and derivative, the initial condition $y(1) = y_0$ is incorporated as a constant term here. Thus,

$$ y(x) = y_0 + \lambda {}^{H}I^{\alpha}_{q,\tau} y(x) + ^{H} I^{\alpha}_{q,\tau} f(x). $$

Step 2: Define an operator and apply fixed point theorem. Define an operator $\mathcal{T}$ acting on the space $C([1, M])$, for arbitrary $M > 1$, by

$$ (\mathcal{T}y)(x) = y_0 + \lambda {}^{H}I^{\alpha}_{q,\tau} y(x) + ^{H} I^{\alpha}_{q,\tau} f(x). $$

Our goal is to show $\mathcal{T}$ has a unique fixed point in $C([1, M])$, i.e. a unique solution to

$$ y = \mathcal{T}y. $$

Step 3: Show $\mathcal{T}$ is a contraction on a suitable complete metric space. Using the definition of ${}^{H}I^{\alpha}_{q,\tau}$, for any $y_1, y_2 \in C([1, M])$,

$$ |(\mathcal{T}y_1)(x) - (\mathcal{T}y_2)(x)| = |\lambda| \left|{}^{H}I^{\alpha}_{q,\tau}(y_1 - y_2)(x)\right|. $$

By the positivity of the kernel and continuity of functions,

$$ \left|{}^{H}I^{\alpha}_{q,\tau}(y_1 - y_2)(x)\right| \leqslant \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_1^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{|y_1(t) - y_2(t)|}{t} dt. $$

Utilizing the supremum norm $\|y\| = \max_{t\in[1,M]} |y(t)|$,

$$|(\mathcal{T}y_1)(x) - (\mathcal{T}y_2)(x)| \leqslant |\lambda| \frac{\|y_1 - y_2\|}{\Gamma_{q,\tau}(\alpha)} \int_1^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{dt}{t}.$$

Make substitution $u = \log \frac{x}{t}$, $dt/t = -du$, limits change from $t = 1$ to $t = x$ into $u = \log x$ down to 0, so

$$\int_1^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{dt}{t} = \int_0^{\log x} u^{\alpha-1} du = \frac{(\log x)^\alpha}{\alpha}.$$

Hence, we get

$$|(\mathcal{T}y_1)(x) - (\mathcal{T}y_2)(x)| \leqslant |\lambda| \frac{(\log x)^\alpha}{\alpha\Gamma_{q,\tau}(\alpha)} \|y_1 - y_2\|.$$

Taking supremum over $x \in [1, M]$:

$$\|\mathcal{T}y_1 - \mathcal{T}y_2\| \leqslant |\lambda| \frac{(\log M)^\alpha}{\alpha\Gamma_{q,\tau}(\alpha)} \|y_1 - y_2\|.$$

Step 4: Contraction mapping condition: Choose M sufficiently close to 1 so that

$$L := |\lambda| \frac{(\log M)^\alpha}{\alpha\Gamma_{q,\tau}(\alpha)} < 1.$$

Then $\mathcal{T}$ is a contraction on $C([1, M])$. By Banach fixed point theorem, $\mathcal{T}$ has a unique fixed point y on $[1, M]$.

Step 5: Extension to $[1, \infty)$: The solution on $[1, M]$ can be extended stepwise to the entire domain $[1, \infty)$ by repeating the contraction argument on intervals $[M, M']$, ensuring existence and uniqueness on the whole domain.

Step 6: Explicit solution The fixed point satisfies

$$y(x) = y_0 + \lambda {}^H I^\alpha_{q,\tau} y(x) + {}^H I^\alpha_{q,\tau} f(x).$$

By successive approximations and properties of the generalized Mittag-Leffler function, the explicit solution is

$$y(x) = y_0 E_\alpha^{(q,\tau)}\left(\lambda(\log x)^\alpha\right) + {}^H I^\alpha_{q,\tau} f(x).$$

This completes the proof. □

3.1. Nonlinear (q, τ)-Hadamard Fractional Problem

Consider the nonlinear initial value problem

$$^H D^\alpha_{q,\tau} y(x) = F(x, y(x)), \quad y(1) = y_0, \tag{3.1}$$

where $^H D^\alpha_{q,\tau}$ is the (q, τ)-Hadamard fractional derivative of order $\alpha > 0$, and

$$F : [1, \infty) \times \mathbb{R} \to \mathbb{R}$$

is a continuous nonlinear function. Utilizing the (q, τ)-Hadamard fractional integral operator $^H I^\alpha_{q,\tau}$, the equation can be reformulated as

$$y(x) = y_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_1^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{F(t, y(t))}{t} dt.$$

Theorem 3.2. *Consider Eq.(3.1). Suppose* F *satisfies:*

- ***Lipschitz condition:*** *There exists* $L > 0$ *such that*
$$|F(x,u) - F(x,v)| \leqslant L|u - v|, \quad \forall x \in [1, M], \quad u, v \in \mathbb{R}.$$
- $F(\cdot, y(\cdot))$ *is continuous and bounded on* $[1, M]$ *for any continuous function* y.

Then, there exists $M > 1$ *sufficiently close to 1 such that the integral operator*
$$(\mathcal{T}y)(x) := y_0 + {}^{H}I^{\alpha}_{q,\tau} F(x, y(x))$$
is a contraction on $C([1, M])$*, ensuring a unique fixed point* $y \in C([1, M])$*, which is the unique solution to the nonlinear problem on* $[1, M]$*. By extending the argument stepwise, existence and uniqueness hold on* $[1, \infty)$.

Proof. The proof follows from the contraction mapping principle:
$$\begin{aligned}|(\mathcal{T}y_1)(x) - (\mathcal{T}y_2)(x)| &\leqslant \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_1^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{|F(t, y_1(t)) - F(t, y_2(t))|}{t} dt \\ &\leqslant L\|y_1 - y_2\| \frac{(\log x)^\alpha}{\alpha \Gamma_{q,\tau}(\alpha)}.\end{aligned}$$
Choosing M so that
$$L \frac{(\log M)^\alpha}{\alpha \Gamma_{q,\tau}(\alpha)} < 1,$$
we get a contraction, hence a unique fixed point. □

Example 3.3. Example Satisfying the Inequality Condition: We aim to satisfy the condition
$$L \cdot \frac{(\log M)^\alpha}{\alpha \, \Gamma_{q,\tau}(\alpha)} < 1,$$
where: $L > 0$ is the Lipschitz constant, $M > 1$ is the upper limit of the logarithmic domain, $0 < \alpha < 1$ is the order of the fractional integral, and $\Gamma_{q,\tau}(\alpha)$ is the generalized (q, τ)-Gamma function. We choose the following values:
$$\alpha = 0.5, \quad q = 0.5, \quad \tau = 1, \quad L = 0.1, \quad M = e^2 \Rightarrow \log M = 2.$$
Now compute:
$$L \times \frac{(\log M)^\alpha}{\alpha \, \Gamma_{q,\tau}(\alpha)} = 0.1 \times \frac{2^{0.5}}{0.5 \cdot \Gamma_{0.5,1}(0.5)}.$$
Using a numerical approximation: $\Gamma_{0.5,1}(0.5) \approx 1.222$, so we get:
$$0.1 \times \frac{1.4142}{0.611} \approx 0.1 \times 2.315 \approx 0.2315 < 1.$$
The inequality is satisfied for the given values, confirming that the condition required for existence theorems involving (q, τ)-Hadamard-type operators holds.
Now, we let $\tau = 2$. Consider the inequality
$$L \times \frac{(\log M)^\alpha}{\alpha \, \Gamma_{q,\tau}(\alpha)} < 1,$$

with parameters:

$$\alpha = 0.6, \quad q = 0.5, \quad \tau = 2, \quad L = 0.08, \quad M = e^3.$$

Then,

$$L \times \frac{(\log M)^\alpha}{\alpha\, \Gamma_{0.5,2}(\alpha)} = 0.08 \times \frac{3^{0.6}}{0.6 \times \Gamma_{0.5,2}(0.6)}.$$

Assuming approximations:

$$3^{0.6} \approx 2.049, \quad \Gamma_{0.5,2}(0.6) \approx 1.73,$$

we obtain

$$0.08 \times \frac{2.049}{1.038} \approx 0.1579 < 1.$$

This confirms the inequality holds for the selected parameters.

Theorem 3.4 (Existence for Nonlinear (q,τ)-Hadamard Systems with $f(x,y,y')$)**.** *Let $\alpha \in (0,1]$, and consider the system of fractional differential equations:*

$${}^{H}D^{\alpha}_{q,\tau} y_i(x) = f_i\left(x, \vec{y}(x), \vec{y}'(x)\right), \quad i = 1,\dots,m, \quad x \in [a,b],$$

with initial conditions

$${}^{H}I^{1-\alpha}_{q,\tau} y_i(x)\big|_{x=a} = y_{i,0}, \quad i = 1,\dots,m.$$

Suppose the following hold

1. *Each $f_i(x,\vec{y},\vec{y}')$ is continuous on $[a,b] \times \mathbb{R}^m \times \mathbb{R}^m$,*
2. *Each f_i satisfies a Lipschitz condition in $\vec{y},\vec{y}'$ uniformly in x:*

$$|f_i(x,\vec{y}_1,\vec{z}_1) - f_i(x,\vec{y}_2,\vec{z}_2)| \leqslant L_1\|\vec{y}_1 - \vec{y}_2\| + L_2\|\vec{z}_1 - \vec{z}_2\|,$$

3. *f_i is bounded on bounded subsets: $|f_i(x,\vec{y},\vec{z})| \leqslant M$,*

Then, the system has at least one solution $\vec{y} \in C^1([a,b];\mathbb{R}^m)$.

Proof. Let $\mathcal{B} := \{\vec{y} \in C^1([a,b];\mathbb{R}^m)\}$ with norm

$$\|\vec{y}\| := \sup_{x\in[a,b]} \|\vec{y}(x)\| + \sup_{x\in[a,b]} \|\vec{y}'(x)\|.$$

Define an operator $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ by

$$(\mathcal{T}\vec{y})(x) := \vec{y}_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_a^x \left(\log\frac{x}{\xi}\right)^{\alpha-1} \frac{\vec{f}(\xi,\vec{y}(\xi),\vec{y}'(\xi))}{\xi}\, d\xi.$$

We show $\mathcal{T}$ is a contraction on a closed ball in $\mathcal{B}$. Let $\vec{y}_1, \vec{y}_2 \in \mathcal{B}$. Then for each $x \in [a,b]$:

$$\begin{aligned}
\|\mathcal{T}\vec{y}_1(x) - \mathcal{T}\vec{y}_2(x)\| &\leqslant \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_a^x \left(\log\frac{x}{\xi}\right)^{\alpha-1} \frac{\|f(\xi,\vec{y}_1(\xi),\vec{y}_1'(\xi)) - f(\xi,\vec{y}_2(\xi),\vec{y}_2'(\xi))\|}{\xi}\, d\xi \\
&\leqslant \frac{(L_1\|\vec{y}_1 - \vec{y}_2\| + L_2\|\vec{y}_1' - \vec{y}_2'\|)}{\Gamma_{q,\tau}(\alpha)} \int_a^x \left(\log\frac{x}{\xi}\right)^{\alpha-1} \frac{1}{\xi}\, d\xi \\
&\leqslant \frac{(L_1 + L_2)}{\Gamma_{q,\tau}(\alpha)} (\log(b/a))^\alpha \|\vec{y}_1 - \vec{y}_2\|.
\end{aligned}$$

Let $K := \dfrac{(L_1 + L_2)(\log(b/a))^{\alpha}}{\Gamma_{q,\tau}(\alpha)}$. If $K < 1$, then $\mathcal{T}$ is a contraction. By the Banach fixed-point theorem, $\mathcal{T}$ has a unique fixed point $\vec{y} \in \mathcal{B}$, which satisfies the system. □

As an application of Theorem 3.4, we have the following consequence:

Corollary 3.5 (Existence for Fractional Lozi System with (q,τ)-Hadamard Derivative). *Let $a, b \in \mathbb{R}$, and let $0 < \alpha \leqslant 1$. Consider the system*

$$\begin{cases} {}^{H}D^{\alpha}_{q,\tau}x(t) = 1 - a|x(t)| + y(t), \\ {}^{H}D^{\alpha}_{q,\tau}y(t) = bx(t), \end{cases} \qquad t \in [t_0, T],$$

with initial conditions

$${}^{H}I^{1-\alpha}_{q,\tau}x(t)\big|_{t=t_0} = x_0, \quad {}^{H}I^{1-\alpha}_{q,\tau}y(t)\big|_{t=t_0} = y_0.$$

Then this system admits at least one solution $(x(t), y(t)) \in C^1([t_0, T])$ provided $\log(T/t_0)^{\alpha} < \frac{\Gamma_{q,\tau}(\alpha)}{L}$, where $L = \max\{a, b\} + 1$.

Theorem 3.6 (Existence of Solutions for a Generalized 3D (q,τ)-Hadamard Fractional System)**.** *Let $f_1, f_2, f_3 : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ be continuous functions satisfying:*

1. *Lipschitz condition: for all $t \in [a, b]$ and for all $(x_i, y_i, z_i) \in \mathbb{R}^3$,*

$$|f_j(t, x_1, y_1, z_1) - f_j(t, x_2, y_2, z_2)| \leqslant L\left(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\right),$$

   for $j = 1, 2, 3$,
2. *and boundedness: $|f_j(t, x, y, z)| \leqslant M$, for all $(t, x, y, z) \in [a, b] \times \mathbb{R}^3$.*

Then the initial value problem:

$$\begin{cases} {}^{H}D^{\alpha}_{q,\tau}x(t) = f_1(t, x(t), y(t), z(t)), \\ {}^{H}D^{\alpha}_{q,\tau}y(t) = f_2(t, x(t), y(t), z(t)), \\ {}^{H}D^{\alpha}_{q,\tau}z(t) = f_3(t, x(t), y(t), z(t)), \\ {}^{H}D^{-\alpha}_{q,\tau}x(t)|_{t=a} = x_0, \\ {}^{H}D^{-\alpha}_{q,\tau}y(t)|_{t=a} = y_0, \\ {}^{H}D^{-\alpha}_{q,\tau}z(t)|_{t=a} = z_0, \end{cases}$$

has a unique continuous solution on $[a, b]$.

Proof. Define the operator $\mathcal{T} : C([a, b])^3 \to C([a, b])^3$ by

$$\mathcal{T}(x, y, z)(t) = \left(x_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} f_1(s, x(s), y(s), z(s)) \frac{ds}{s}, \ldots \right).$$

We show that $\mathcal{T}$ is a contraction on the complete metric space $C([a, b])^3$ under the sup norm:

$$\|(x, y, z) - (\tilde{x}, \tilde{y}, \tilde{z})\| \leqslant \frac{L(b-a)^{\alpha}}{\Gamma_{q,\tau}(1+\alpha)} \|(x - \tilde{x}, y - \tilde{y}, z - \tilde{z})\|.$$

If $L(b - a)^{\alpha}/\Gamma_{q,\tau}(1 + \alpha) < 1$, then by Banach's fixed-point theorem, $\mathcal{T}$ has a unique fixed point, which is the unique solution of the system. □

As an application, we have the following result:

Theorem 3.7 (Synchronization via Linear Feedback Control)**.** *Let* $(x_m(t), y_m(t), z_m(t))$ *be the solution of the master system (Theorem 3.6)*

$$\begin{cases} {}^{H}D^{\alpha}_{q,\tau} x_m(t) = 1 - a|x_m(t)| + y_m(t), \\ {}^{H}D^{\alpha}_{q,\tau} y_m(t) = b x_m(t) + z_m(t), \\ {}^{H}D^{\alpha}_{q,\tau} z_m(t) = -c y_m(t), \end{cases}$$

and let $(x_s(t), y_s(t), z_s(t))$ *satisfy the controlled slave system:*

$$\begin{cases} {}^{H}D^{\alpha}_{q,\tau} x_s(t) = 1 - a|x_s(t)| + y_s(t) + u_1(t), \\ {}^{H}D^{\alpha}_{q,\tau} y_s(t) = b x_s(t) + z_s(t) + u_2(t), \\ {}^{H}D^{\alpha}_{q,\tau} z_s(t) = -c y_s(t) + u_3(t), \end{cases}$$

with control laws:

$$u_1(t) = -k_1 e_1(t), \quad u_2(t) = -k_2 e_2(t), \quad u_3(t) = -k_3 e_3(t),$$

where $e_i(t) = x_s(t) - x_m(t)$*, etc., and* $k_i > 0$ *are constants. Then for* $0 < \alpha \leqslant 1$*, the error vector* $\vec{e}(t) = (e_1(t), e_2(t), e_3(t))$ *tends to zero as* $t \to \infty$*, i.e., the slave synchronizes with the master.*

Proof. Define the error system

$$\begin{cases} {}^{H}D^{\alpha}_{q,\tau} e_1(t) = -a(|x_s| - |x_m|) + e_2(t) - k_1 e_1(t), \\ {}^{H}D^{\alpha}_{q,\tau} e_2(t) = b e_1(t) + e_3(t) - k_2 e_2(t), \\ {}^{H}D^{\alpha}_{q,\tau} e_3(t) = -c e_2(t) - k_3 e_3(t). \end{cases}$$

Define a Lyapunov candidate

$$V(t) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2\right).$$

Then,

$${}^{H}D^{\alpha}_{q,\tau} V(t) = e_1 {}^{H}D^{\alpha}_{q,\tau} e_1 + e_2 {}^{H}D^{\alpha}_{q,\tau} e_2 + e_3 {}^{H}D^{\alpha}_{q,\tau} e_3.$$

Substituting the system

$$\begin{aligned} {}^{H}D^{\alpha}_{q,\tau} V(t) = {} & e_1\left(-a(|x_s| - |x_m|) + e_2 - k_1 e_1\right) + e_2\left(b e_1 + e_3 - k_2 e_2\right) \\ & + e_3\left(-c e_2 - k_3 e_3\right). \end{aligned}$$

Using $\big||x_s| - |x_m|\big| \leqslant |e_1|$:

$$-a(|x_s| - |x_m|) e_1 \leqslant a e_1^2.$$

Then, we have

$$\begin{aligned} {}^{H}D^{\alpha}_{q,\tau} V(t) & \leqslant a e_1^2 + e_1 e_2 - k_1 e_1^2 + b e_1 e_2 + e_2 e_3 - k_2 e_2^2 - c e_2 e_3 - k_3 e_3^2 \\ & = (-k_1 + a) e_1^2 + (1 + b) e_1 e_2 + (1 - c) e_2 e_3 - k_2 e_2^2 - k_3 e_3^2. \end{aligned}$$

Apply Young's inequality:

$$(1+b)e_1e_2 \leqslant \frac{(1+b)^2}{2\delta_1}e_1^2 + \frac{\delta_1}{2}e_2^2, \quad (1-c)e_2e_3 \leqslant \frac{(1-c)^2}{2\delta_2}e_2^2 + \frac{\delta_2}{2}e_3^2.$$

Then, we get

$$^{H}D^{\alpha}_{q,\tau}V(t) \leqslant \left(-k_1 + a + \frac{(1+b)^2}{2\delta_1}\right)e_1^2 + \left(-k_2 + \frac{\delta_1}{2} + \frac{(1-c)^2}{2\delta_2}\right)e_2^2 + \left(-k_3 + \frac{\delta_2}{2}\right)e_3^2.$$

Choose δ_1, δ_2 small and k_1, k_2, k_3 large enough so that all coefficients are negative. Then:

$$^{H}D^{\alpha}_{q,\tau}V(t) \leqslant -\lambda V(t), \quad \lambda > 0.$$

By the comparison principle and properties of the (q,τ)-Mittag-Leffler function:

$$V(t) \leqslant V(0)E_{\alpha}^{(q,\tau)}(-\lambda(\log t)^{\alpha}) \to 0 \quad \text{as } t \to \infty.$$

Therefore, $e_i(t) \to 0$, and synchronization is achieved. □

Algorithm, Simulation and Stability of 3D (q,τ)-Hadamard Fractional System is given as follows:

Algorithm 3.8 (H)**.** [3D (q,τ)-Hadamard Fractional System Simulation]
Require Fractional order $\alpha \in (0,1]$, deformation parameters $q,\tau > 0$, nonlinear functions f_1, f_2, f_3, initial data x_0, y_0, z_0, final time T, time step Δt
Ensure Approximations $x(t), y(t), z(t)$ on $[a,b]$
State Discretize time domain: $t_n = a + n\Delta t$, for $n = 0, 1, \ldots, N$, where $N = T/\Delta t$
State Initialize: $x[0] \leftarrow x_0$, $y[0] \leftarrow y_0$, $z[0] \leftarrow z_0$
For$n = 1$ to N
State Compute Hadamard-type weights:

$$w_j^{(\alpha)} = \frac{\left(\log \frac{t_n}{t_j}\right)^{\alpha-1}}{\Gamma_{q,\tau}(\alpha)} \cdot \frac{1}{t_j}, \quad j = 0, 1, \ldots, n-1$$

State Update solution via fractional Euler rule:

$$x[n] \leftarrow x_0 + \sum_{j=0}^{n-1} w_j^{(\alpha)} f_1(t_j, x[j], y[j], z[j]) \cdot \Delta t,$$

$$y[n] \leftarrow y_0 + \sum_{j=0}^{n-1} w_j^{(\alpha)} f_2(t_j, x[j], y[j], z[j]) \cdot \Delta t,$$

$$z[n] \leftarrow z_0 + \sum_{j=0}^{n-1} w_j^{(\alpha)} f_3(t_j, x[j], y[j], z[j]) \cdot \Delta t$$

EndFor State (Optional) Evaluate Lyapunov function:

$$V(t_n) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2)$$

State Check decay:

$${}^{H}D^{\alpha}_{q,\tau}V(t) \leqslant -\lambda V(t) \Rightarrow \text{stable synchronization}$$

State Return Arrays $\{x[n]\}, \{y[n]\}, \{z[n]\}$

Lemma 3.9 (Grönwall Inequality in the (q,τ)-Hadamard Calculus)**.** *Let $u(x)$ be a non-negative, continuous function on $[1,b]$, and suppose that:*

$$u(x) \leqslant A + B \cdot {}^{H} I^{\alpha}_{q,\tau} u(x),$$

for constants $A, B \geqslant 0$ and $0 < \alpha < 1$, where

$${}^{H}I^{\alpha}_{q,\tau}f(x) = \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_1^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t}\, dt.$$

Then

$$u(x) \leqslant A \cdot E_{\alpha}^{(q,\tau)}\left(B(\log x)^{\alpha}\right).$$

Proof. We define a sequence of approximations:

$$u_0(x) := A, \quad u_{n+1}(x) := A + B \cdot {}^{H} I^{\alpha}_{q,\tau} u_n(x), \quad n \geqslant 0.$$

Step 1: Show $u_n(x) \leqslant u_{n+1}(x)$: Since $u_0(x) = A$, we have:

$$\begin{aligned} u_1(x) &= A + B \cdot {}^{H} I^{\alpha}_{q,\tau} A = A + \frac{AB}{\Gamma_{q,\tau}(\alpha)} \int_1^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{1}{t} dt \\ &= A + AB \cdot \frac{(\log x)^{\alpha}}{\alpha \Gamma_{q,\tau}(\alpha)}. \end{aligned}$$

Similarly, all $u_n(x)$ are non-decreasing in n.

Step 2: Prove by induction: Assume

$$u_n(x) \leqslant A \sum_{k=0}^{n} \frac{[B(\log x)^{\alpha}]^k}{\Gamma_{q,\tau}(\alpha k + 1)}.$$

Then, we have

$$\begin{aligned} u_{n+1}(x) &= A + B \cdot {}^{H} I^{\alpha}_{q,\tau} u_n(x) \\ &\leqslant A + \frac{B}{\Gamma_{q,\tau}(\alpha)} \int_1^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{1}{t} \cdot A \sum_{k=0}^{n} \frac{[B(\log t)^{\alpha}]^k}{\Gamma_{q,\tau}(\alpha k + 1)} dt \\ &= A + A \sum_{k=0}^{n} \frac{B^{k+1}}{\Gamma_{q,\tau}(\alpha k + 1)} \cdot \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_1^x \left(\log \frac{x}{t}\right)^{\alpha-1} (\log t)^{\alpha k} \frac{dt}{t}. \end{aligned}$$

Now, by changing the variables $s = \log t$, then $\log x = \log x - \log t + \log t = \log x$, and via convolution-type structure of Hadamard integrals, the integral becomes, as follows:

$$\int_0^{\log x} (\log x - s)^{\alpha-1} s^{\alpha k} ds = (\log x)^{\alpha(k+1)} \cdot \frac{\Gamma(\alpha)\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}.$$

Thus, we obtain

$$u_{n+1}(x) \leqslant A \sum_{k=0}^{n+1} \frac{[B(\log x)^{\alpha}]^k}{\Gamma_{q,\tau}(\alpha k+1)}.$$

Taking the limit $n \to \infty$, we get

$$u(x) \leqslant \lim_{n\to\infty} u_n(x) \leqslant A \sum_{k=0}^{\infty} \frac{[B(\log x)^{\alpha}]^k}{\Gamma_{q,\tau}(\alpha k+1)}$$

$$= A \cdot E_{\alpha}^{(q,\tau)}\left(B(\log x)^{\alpha}\right).$$

□

Lemma 3.10 ((q,τ)-Gronwall Inequality: special case)**.** *Let* $\alpha \in (0,1)$*, and suppose that* $\phi, \psi \in C([a,T], \mathbb{R}_+)$ *satisfy the inequality*

$$\phi(t) \leqslant \psi(t) + B \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{\phi(\tau)}{\tau}\, d\tau,$$

for all $t \in [a,T]$*, where* $B > 0$ *is a constant. Then*

$$\phi(t) \leqslant \psi(t) + B \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{\psi(\tau)}{\tau}\, E_{\alpha}^{(q,\tau)}\left(B\left(\log \frac{t}{\tau}\right)^{\alpha}\right) d\tau.$$

In particular, if $\psi(t) = C(\log \frac{t}{a})^{\gamma}$*, then*

$$\phi(t) \leqslant C(\log \tfrac{t}{a})^{\gamma}\, E_{\alpha}^{(q,\tau)}\left(B(\log \tfrac{t}{a})^{\alpha}\right).$$

Proof. We proceed by iterative substitution (Picard iteration). Define the operator

$$(\mathcal{H}\phi)(t) := B \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{\phi(\tau)}{\tau}\, d\tau.$$

Then the inequality becomes

$$\phi(t) \leqslant \psi(t) + (\mathcal{H}\phi)(t).$$

We define a sequence $\{\phi_n(t)\}$ by:

$$\phi_0(t) = \psi(t), \quad \phi_{n+1}(t) = \psi(t) + \mathcal{H}\phi_n(t).$$

By induction

$$\phi_1(t) = \psi(t) + \mathcal{H}\psi(t), \quad \phi_2(t) = \psi(t) + \mathcal{H}\psi(t) + \mathcal{H}^2\psi(t), \quad \text{etc.}$$

Hence, we get

$$\phi_n(t) = \psi(t) + \sum_{k=1}^{n} \mathcal{H}^k \psi(t).$$

Assuming convergence, we define:

$$\phi(t) \leqslant \psi(t) + \sum_{k=1}^{\infty} \mathcal{H}^k \psi(t).$$

Let us now estimate the k-fold convolution kernel:

$$\mathcal{H}^k \psi(t) = B^k \int_a^t \cdots \int_a^{\tau_{k-1}} \prod_{j=1}^{k} \left(\log \frac{\tau_{j-1}}{\tau_j} \right)^{\alpha-1} \frac{\psi(\tau_k)}{\tau_1 \cdots \tau_k} d\tau_k \cdots d\tau_1,$$

with $\tau_0 := t$. Let $\psi(\tau_k) \leqslant \psi(t)$ (monotonicity or uniform bound), then

$$\mathcal{H}^k \psi(t) \leqslant \psi(t) \cdot \frac{B^k (\log \frac{t}{a})^{\alpha k}}{\Gamma_{q,\tau}(\alpha k + 1)}.$$

Thus, we obtain

$$\phi(t) \leqslant \psi(t) \left(1 + \sum_{k=1}^{\infty} \frac{(B(\log \frac{t}{a})^{\alpha})^k}{\Gamma_{q,\tau}(\alpha k + 1)} \right) = \psi(t) \cdot E_{\alpha}^{(q,\tau)} \left(B(\log \tfrac{t}{a})^{\alpha} \right).$$

Now consider the general bound:

$$\phi(t) \leqslant \psi(t) + \mathcal{H} \left(\psi(t) \cdot E_{\alpha}^{(q,\tau)} (B(\log \tfrac{t}{a})^{\alpha}) \right),$$

and exchange the factor since it is independent of the integration variable τ, yielding:

$$\phi(t) \leqslant \psi(t) + B \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} \frac{\psi(\tau)}{\tau} E_{\alpha}^{(q,\tau)} \left(B(\log \frac{t}{\tau})^{\alpha} \right) d\tau,$$

which completes the proof. □

Lemma 3.11. *Suppose* $u(x) \leqslant A + B \cdot^{H} I_{q,\tau}^{\alpha}[u(x)^{\rho}]$, *where* $\rho > 1$. *Then there exists a function* $v(x)$ *such that:*

$$v(x) = A + B \cdot^{H} I_{q,\tau}^{\alpha}[v(x)^{\rho}], \quad u(x) \leqslant v(x).$$

Proof. Define an operator

$$(Tv)(x) := A + B \cdot^{H} I_{q,\tau}^{\alpha}[v(x)^{\rho}].$$

Under boundedness assumptions, T maps a convex, compact subset of $C^{+}([1,b])$ into itself and is continuous. Hence, by Schauder's theorem, T has a fixed point. □

Example 3.12. Let $A = 1, B = 0.2, \rho = 2, \alpha = 0.5$. Use iteration:

$$v_0(x) = 1, \quad v_{n+1}(x) = 1 + \frac{0.2}{\Gamma_{q,\tau}(0.5)} \int_1^x \left(\log \frac{x}{t} \right)^{-0.5} \frac{v_n(t)^2}{t} dt.$$

This sequence converges to the minimal solution $v(x)$, which bounds $u(x)$ from above.

4. Stability of Nonlinear (q,τ)-Hadamard Fractional Equations

Definition 4.1 (Stability concept)**.** Consider the nonlinear fractional differential equation involving the (q,τ)-Hadamard operator:

$$ {}^{H}D^{\alpha}_{q,\tau}y(x) = F(x,y(x)), \quad y(1) = y_0, $$

where $0 < \alpha < 1$, and $F : [1,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies appropriate conditions. The solution $y(x)$ is said to be **stable** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that any other solution $\tilde{y}(x)$ satisfying $|\tilde{y}(1) - y(1)| < \delta$ also satisfies

$$ |y(x) - \tilde{y}(x)| < \varepsilon, \quad \forall x \in [1,T], $$

for some fixed $T > 1$.

Suppose $F(x,y)$ satisfies a Lipschitz condition:

$$ |F(x,u) - F(x,v)| \leqslant L|u-v|, \quad \forall u,v \in \mathbb{R}, \quad x \in [1,T]. $$

Let $y(x)$ and $\tilde{y}(x)$ be two solutions with initial values $y(1) = y_0$, $\tilde{y}(1) = \tilde{y}_0$. Then the difference satisfies

$$ |y(x) - \tilde{y}(x)| \leqslant |y_0 - \tilde{y}_0| + L \cdot I^{\alpha}_{H;q,\tau}|y-\tilde{y}|(x). $$

Define

$$ e(x) := |y(x) - \tilde{y}(x)|. $$

Then, we have

$$ e(x) \leqslant \delta + L \cdot \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_1^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{e(t)}{t} dt. $$

Using a (q,τ)-version of Grönwall's inequality (Lemma 3.10), we obtain

$$ e(x) \leqslant \delta E^{(q,\tau)}_{\alpha}\left(L(\log x)^{\alpha}\right), $$

which shows that the solution depends continuously on the initial value y_0, hence the system (3.1) is stable. Moreover, if $F(x,y)$ is bounded, say $|F(x,y)| \leqslant M$, then from the integral form:

$$ |y(x)| \leqslant |y_0| + \frac{M}{\Gamma_{q,\tau}(\alpha)} \int_1^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{1}{t} dt = |y_0| + \frac{M(\log x)^{\alpha}}{\alpha\Gamma_{q,\tau}(\alpha)}. $$

So $y(x)$ grows at most like $(\log x)^{\alpha}$, i.e., slowly as $x \to \infty$. More refined asymptotics can be derived depending on the structure of F, especially if it has polynomial or exponential behavior in x or y. The nonlinear (q,τ)-Hadamard fractional differential equations exhibit continuous dependence on initial conditions and stability under Lipschitz assumptions. Their solutions are generally slowly growing and bounded under mild conditions on the nonlinearity F.

Theorem 4.2 (Ulam–Hyers Stability)**.** *Let* $u : [a,T] \to \mathbb{R}$ *satisfy the nonlinear* (q,τ)*-Hadamard fractional initial value problem:*

$$^{H}D^{\alpha}_{q,\tau}u(t) = f(t,u(t)), \quad u(a) = u_0, \quad 0 < \alpha < 1,$$

where $^{H}D^{\alpha}_{q,\tau}$ *is the left-sided* (q,τ)*-Hadamard fractional derivative of order* $\alpha \in (0,1)$*, and* $f : [a,T] \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies the Lipschitz condition:*

$$|f(t,u_1) - f(t,u_2)| \leqslant L|u_1 - u_2|, \quad \forall t \in [a,T],\ u_1,u_2 \in \mathbb{R}.$$

Assume that

$$\frac{(\log M)^{\alpha}}{\alpha\,\Gamma_{q,\tau}(\alpha)} \cdot L < 1, \quad \textit{where } M := \frac{T}{a}.$$

Then the equation is Ulam–Hyers stable: for every function $v(t)$ *satisfying*

$$\left|^{H}D^{\alpha}_{q,\tau}v(t) - f(t,v(t))\right| \leqslant \varepsilon,$$

there exists a true solution $u(t)$ *such that*

$$|v(t) - u(t)| \leqslant C\varepsilon, \quad \forall t \in [a,T],$$

for some constant $C > 0$ *depending on* $\alpha, L, T, \Gamma_{q,\tau}(\alpha)$*.*

Proof. Apply the (q,τ)-Hadamard fractional integral $^{H}I^{\alpha}_{q,\tau}$ to both sides of the differential equation:

$$u(t) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau,u(\tau))}{\tau}\, d\tau.$$

Let $v(t)$ be an approximate solution satisfying:

$$\left|^{H}D^{\alpha}_{q,\tau}v(t) - f(t,v(t))\right| \leqslant \varepsilon.$$

Then, we get

$$v(t) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau,v(\tau))}{\tau}\, d\tau + e(t),$$

where

$$|e(t)| \leqslant \frac{\varepsilon}{\Gamma_{q,\tau}(\alpha)} \int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{1}{\tau}\, d\tau = \frac{\varepsilon}{\Gamma_{q,\tau}(\alpha)} \cdot \frac{(\log\frac{t}{a})^{\alpha}}{\alpha}.$$

Define $\phi(t) = |v(t) - u(t)|$. Then, we have

$$\begin{aligned}\phi(t) &= \left|\frac{1}{\Gamma_{q,\tau}(\alpha)} \int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau,v(\tau)) - f(\tau,u(\tau))}{\tau}\, d\tau + e(t)\right| \\ &\leqslant \frac{L}{\Gamma_{q,\tau}(\alpha)} \int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{\phi(\tau)}{\tau}\, d\tau + \frac{\varepsilon}{\Gamma_{q,\tau}(\alpha)} \cdot \frac{(\log\frac{t}{a})^{\alpha}}{\alpha}.\end{aligned}$$

Apply the (q,τ)-Gronwall inequality (Lemma 3.10), for the case

$$A(t) := \frac{\varepsilon}{\alpha\Gamma_{q,\tau}(\alpha)}(\log\frac{t}{a})^{\alpha}, \quad B := \frac{L}{\Gamma_{q,\tau}(\alpha)}.$$

Thus, we obtain

$$\phi(t) \leqslant \frac{\varepsilon}{\alpha\Gamma_{q,\tau}(\alpha)}(\log\frac{t}{a})^{\alpha} \cdot E_{\alpha}^{(q,\tau)}\left(\frac{L}{\Gamma_{q,\tau}(\alpha)}(\log\frac{t}{a})^{\alpha}\right).$$

Under the assumption:

$$\frac{L}{\Gamma_{q,\tau}(\alpha)}(\log\frac{T}{a})^{\alpha} < \alpha,$$

the Mittag-Leffler function $E_{\alpha}(\cdot)$ is bounded on $[a,T]$. Therefore, we get

$$|v(t) - u(t)| \leqslant C\varepsilon, \quad \forall t \in [a,T],$$

for some constant $C > 0$. This completes the proof. □

5. Discrete Hadamard Fractional Difference

Let $f : \mathbb{N} \to \mathbb{R}$. Then the (q,τ)-Hadamard fractional difference of order $\alpha > 0$ is defined as:

$${}^{H}\Delta_{q,\tau}^{\alpha} f(n) := \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=1}^{n} \left(\log\frac{n}{k}\right)^{-\alpha} \frac{f(k)}{k}.$$

The corresponding fractional sum (integral) is given by:

$${}^{H}\Delta_{q,\tau}^{-\alpha} f(n) := \frac{1}{\Gamma_{q,\tau}(\alpha)} \sum_{k=1}^{n} \left(\log\frac{n}{k}\right)^{\alpha-1} \frac{f(k)}{k}.$$

Semigroup Property can be formulates, as follows:

$${}^{H}\Delta_{q,\tau}^{-\alpha}\left({}^{H}\Delta_{q,\tau}^{-\beta} f\right)(n) = {}^{H}\Delta_{q,\tau}^{-(\alpha+\beta)} f(n), \quad \text{for } \alpha, \beta > 0.$$

Theorem 5.1 (Uniqueness and Stability in Discrete (q,τ)-Hadamard Fractional Difference Equations)**.** *Consider the discrete (q,τ)-Hadamard fractional difference equation:*

$${}^{H}\Delta_{q,\tau}^{\alpha} y(n) = \lambda y(n) + f(n), \quad 0 < \alpha \leqslant 1,$$

with initial condition

$${}^{H}\Delta_{q,\tau}^{-\alpha} y(n)\big|_{n=1} = y_0.$$

Suppose that two solutions $y(n)$ and $z(n)$ satisfy the inequality

$$|y(n) - z(n)| \leqslant C + {}^{H}\Delta_{q,\tau}^{-\alpha}\left[L|y(n) - z(n)|\right],$$

for constants $C \geqslant 0$, $L > 0$. Then:

1. *If $C = 0$, then $y(n) = z(n)$ for all n (uniqueness).*

2. *If* $C > 0$, *then*
$$|y(n) - z(n)| \leqslant C \cdot E_{\alpha}^{(q,\tau)}\left(L(\log n)^{\alpha}\right).$$

Proof. Let $e(n) := |y(n) - z(n)|$. The given inequality implies:

$$e(n) \leqslant C + \frac{1}{\Gamma_{q,\tau}(\alpha)} \sum_{k=1}^{n} \frac{(\log(n/k))^{\alpha-1}}{k} L e(k).$$

Define the following functional:

$$K(n,k) := \frac{(\log(n/k))^{\alpha-1}}{k \cdot \Gamma_{q,\tau}(\alpha)}, \quad \text{so that} \quad e(n) \leqslant C + L \sum_{k=1}^{n} K(n,k) e(k).$$

We proceed by constructing a Picard sequence:

$$e_0(n) := C, \quad e_{m+1}(n) := C + L \sum_{k=1}^{n} K(n,k) e_m(k).$$

We show by induction that:

$$e_m(n) \leqslant C \sum_{j=0}^{m} \frac{(L(\log n)^{\alpha})^j}{\Gamma_{q,\tau}(\alpha j + 1)}.$$

Base case: For $m = 0$, we have $e_0(n) = C$, and the sum becomes 1.

Inductive step: Assume the bound holds for $e_m(n)$. Then:

$$\begin{aligned}
e_{m+1}(n) &= C + L \sum_{k=1}^{n} K(n,k) e_m(k) \\
&\leqslant C + L \sum_{k=1}^{n} K(n,k) \cdot C \sum_{j=0}^{m} \frac{(L(\log k)^{\alpha})^j}{\Gamma_{q,\tau}(\alpha j + 1)} \\
&= C \left(1 + \sum_{j=0}^{m} \frac{L^{j+1}}{\Gamma_{q,\tau}(\alpha j + 1)} \sum_{k=1}^{n} \frac{(\log(n/k))^{\alpha-1} (\log k)^{\alpha j}}{k \cdot \Gamma_{q,\tau}(\alpha)} \right).
\end{aligned}$$

Using $\log(n) = \log(k) + \log(n/k)$, we estimate the sum by bounding $(\log k)^{\alpha j} \leqslant (\log n)^{\alpha j}$, yielding:

$$e_{m+1}(n) \leqslant C \sum_{j=0}^{m+1} \frac{(L(\log n)^{\alpha})^j}{\Gamma_{q,\tau}(\alpha j + 1)}.$$

Taking $m \to \infty$, we obtain:

$$e(n) \leqslant C \sum_{j=0}^{\infty} \frac{(L(\log n)^{\alpha})^j}{\Gamma_{q,\tau}(\alpha j + 1)} = C \cdot E_{\alpha}^{(q,\tau)}(L(\log n)^{\alpha}).$$

If $C = 0$, then $e_0(n) = 0$, so $e_1(n) = 0$, etc., and we conclude:

$$e(n) = 0 \Rightarrow y(n) = z(n), \quad \forall n.$$

□

For example,

$$ {}^{H}\Delta_{q,\tau}^{0.5} y(n) = -y(n), \quad y(1) = 1, \quad \Rightarrow \quad y(n) = 1 - {}^{H}\Delta_{q,\tau}^{-0.5} y(n). $$

Lemma 5.2 (Discrete (q,τ)-Gronwall Inequality)**.** *Let* $\{y(n)\}_{n\in\mathbb{N}}$*,* $\{a(n)\}_{n\in\mathbb{N}}$*, and* $\{b(n)\}_{n\in\mathbb{N}}$ *be sequences of non-negative real numbers. Suppose* $0 < \alpha \leqslant 1$*,* $0 < q < 1$*,* $\tau > 0$*, and that*

$$ y(n) \leqslant a(n) + {}^{H}\Delta_{q,\tau}^{-\alpha} \left[b(n) y(n) \right], \quad \textit{for } n \geqslant 1, $$

where ${}^{H}\Delta_{q,\tau}^{-\alpha}$ *is the discrete Hadamard-type fractional sum defined by*

$$ {}^{H}\Delta_{q,\tau}^{-\alpha}[f(n)] = \frac{1}{\Gamma_{q,\tau}(\alpha)} \sum_{k=1}^{n} (\log(n/k))^{\alpha-1} f(k) \frac{1}{k}. $$

Then, we obtain

$$ y(n) \leqslant a(n) + \sum_{j=1}^{n} \left[\frac{(\log(n/j))^{\alpha-1}}{\Gamma_{q,\tau}(\alpha)} \prod_{i=j}^{n} \left(1 + b(i) \cdot \frac{(\log(i/j))^{\alpha-1}}{\Gamma_{q,\tau}(\alpha)} \right) a(j) \frac{1}{j} \right], $$

and in particular, if $a(n) \leqslant A$ *and* $b(n) \leqslant B$*, then*

$$ y(n) \leqslant A \cdot E_{\alpha}^{(q,\tau)} \left(B (\log n)^{\alpha} \right), $$

where $E_{\alpha}^{(q,\tau)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_{q,\tau}(\alpha k+1)}$ *is the* (q,τ)*-Mittag-Leffler function.*

Proof. We proceed by induction. The inequality

$$ y(n) \leqslant a(n) + \frac{1}{\Gamma_{q,\tau}(\alpha)} \sum_{k=1}^{n} (\log(n/k))^{\alpha-1} b(k) y(k) \frac{1}{k} $$

is of Volterra-type. Iterating this expression gives

$$ y(n) \leqslant a(n) + \frac{1}{\Gamma_{q,\tau}(\alpha)} \sum_{k=1}^{n} (\log(n/k))^{\alpha-1} b(k) \left[a(k) + {}^{H}\Delta_{q,\tau}^{-\alpha}[b(k) y(k)] \right] \frac{1}{k}. $$

Applying the inequality recursively generates a discrete convolution with a kernel involving $(\log(n/k))^{\alpha-1}$, leading to an upper bound that can be expressed as a discrete convolution of $a(k)$ with powers of the kernel scaled by $b(k)$. This iterative form resembles the Volterra series solution and can be bounded above by a discrete version of the (q,τ)-Mittag-Leffler function:

$$ y(n) \leqslant A \cdot \sum_{m=0}^{\infty} \frac{(B(\log n)^{\alpha})^m}{\Gamma_{q,\tau}(\alpha m + 1)} = A \cdot E_{\alpha}^{(q,\tau)} \left(B (\log n)^{\alpha} \right). $$

This completes the proof. □

Theorem 5.3 (Ulam–Hyers Stability for Discrete Nonlinear (q,τ)-Hadamard Fractional Equations)**.** *Consider the nonlinear discrete (q,τ)-Hadamard fractional difference equation:*

$$^{H}\Delta_{q,\tau}^{\alpha} y(n) = f(n, y(n)), \quad 0 < \alpha \leqslant 1,$$

with initial condition:

$$^{H}\Delta_{q,\tau}^{-\alpha} y(n)\big|_{n=1} = y_0.$$

Assume that:

(A1) $f(n,y)$ *is continuous in* y *and satisfies a Lipschitz condition in the second argument:*

$$|f(n,y_1) - f(n,y_2)| \leqslant L|y_1 - y_2|, \quad \forall y_1, y_2 \in \mathbb{R}, \ \forall n.$$

(A2) The fractional solution satisfies the variation-of-constants formula:

$$y(n) = y_0 + {}^{H}\Delta_{q,\tau}^{-\alpha}\left[f(n, y(n))\right].$$

Let $z(n)$ *be an approximate solution satisfying:*

$$\left|{}^{H}\Delta_{q,\tau}^{\alpha} z(n) - f(n, z(n))\right| \leqslant \varepsilon.$$

Then there exists a constant $C > 0$ *such that:*

$$|y(n) - z(n)| \leqslant C \cdot \varepsilon \cdot E_{\alpha}^{(q,\tau)}\left(L(\log n)^{\alpha}\right).$$

Proof. Apply the discrete (q,τ)-Hadamard integral ${}^{H}\Delta_{q,\tau}^{-\alpha}$ to both sides of the equation satisfied by $z(n)$:

$$z(n) = y_0 + {}^{H}\Delta_{q,\tau}^{-\alpha}\left[f(n, z(n)) + \delta(n)\right],$$

where the perturbation $\delta(n) := {}^{H}\Delta_{q,\tau}^{\alpha} z(n) - f(n, z(n))$ satisfies $|\delta(n)| \leqslant \varepsilon$. Hence, we have

$$z(n) = y_0 + {}^{H}\Delta_{q,\tau}^{-\alpha}[f(n, z(n))] + {}^{H}\Delta_{q,\tau}^{-\alpha}[\delta(n)],$$

while the exact solution satisfies:

$$y(n) = y_0 + {}^{H}\Delta_{q,\tau}^{-\alpha}[f(n, y(n))].$$

Taking the difference

$$|y(n) - z(n)| \leqslant \left|{}^{H}\Delta_{q,\tau}^{-\alpha}[f(n, y(n)) - f(n, z(n))]\right| + \left|{}^{H}\Delta_{q,\tau}^{-\alpha}[\delta(n)]\right|.$$

Using the Lipschitz condition and linearity of the fractional sum

$$\left|{}^{H}\Delta_{q,\tau}^{-\alpha}[f(n, y(n)) - f(n, z(n))]\right| \leqslant L \cdot {}^{H}\Delta_{q,\tau}^{-\alpha}[|y(n) - z(n)|],$$

and since $|\delta(n)| \leqslant \varepsilon$, we also have

$$\left|{}^{H}\Delta_{q,\tau}^{-\alpha}[\delta(n)]\right| \leqslant \varepsilon \cdot \frac{(\log n)^{\alpha}}{\alpha \Gamma_{q,\tau}(\alpha)}.$$

Therefore,

$$|y(n) - z(n)| \leqslant L \cdot {}^{H}\Delta_{q,\tau}^{-\alpha}[|y(n) - z(n)|] + \frac{\varepsilon}{\alpha\Gamma_{q,\tau}(\alpha)}(\log n)^{\alpha}.$$

Now define $e(n) := |y(n) - z(n)|$. Then we have

$$e(n) \leqslant \frac{\varepsilon}{\alpha\Gamma_{q,\tau}(\alpha)}(\log n)^{\alpha} + L \cdot {}^{H}\Delta_{q,\tau}^{-\alpha}[e(n)].$$

Apply the discrete (q, τ)-Grönwall inequality. From the Lemma 5.2

$$e(n) \leqslant \frac{\varepsilon}{\alpha\Gamma_{q,\tau}(\alpha)}(\log n)^{\alpha} \cdot E_{\alpha}^{(q,\tau)}(L(\log n)^{\alpha}).$$

Hence, this yields

$$|y(n) - z(n)| \leqslant \frac{\varepsilon}{\alpha\Gamma_{q,\tau}(\alpha)}(\log n)^{\alpha} \cdot E_{\alpha}^{(q,\tau)}(L(\log n)^{\alpha}),$$

which is bounded above by a constant $C > 0$ times $\varepsilon \cdot E_{\alpha}^{(q,\tau)}(L(\log n)^{\alpha})$, completing the proof. □

6. Applications in Quantum Computing

Recent advances in fractional calculus have provided powerful tools for modeling complex dynamical systems with memory, nonlocal feedback, and scale-dependent evolution. The (q, τ)-Hadamard fractional operator introduced in this study offers a flexible and generalized framework to describe such phenomena. In quantum computing, this operator naturally captures decoherence, entanglement decay, and fractional quantum walks with memory, thereby enriching the simulation of open quantum systems and quantum control strategies. Interestingly, similar mathematical structures appear in economic systems where long-term memory, delayed feedback, and anomalous diffusion are fundamental. Applications in macroeconomic growth, optimal control with delays, and market dynamics with persistent volatility can benefit from the same kernel-based modeling approach. By integrating the (q, τ)-deformation into the Hadamard-type fractional calculus, our model unifies these domains through a common analytical lens, enabling interdisciplinary insights into both quantum and economic complexity. The (q, τ)-Hadamard fractional operator provides discrete and nonlocal memory effects that are controlled by the parameters $q \in (0, 1)$ and $\tau > 0$. Several concepts can be contained into these applications (see Table 2).

6.1. Fractional Quantum Walks

Coin and shift operators control quantum walks. One can use a (q, τ)-Hadamard operator as a deformed operator:

$$\mathcal{H}_{q,\tau}^{\alpha} = \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=1}^{n} \left(\log \frac{n}{k}\right)^{-\alpha} \frac{C_k}{k},$$

Table 2: The framework for summarizing the relationship between fractional quantum theory and quantum computation and (q,τ)-Hadamard operators.

Quantum Domain	**Role of (q,τ)-Hadamard**	**Implication**
Quantum Walks	Deformed coin operator	Tunable transition rules
Open Quantum Systems	Memory kernel in evolution	Non-Markovian modeling
Quantum Algebras	Commutation deformation	q-deformed symmetry
Fractional Schrödinger Eq.	Fractional time evolution	Nonlocal quantum dynamics
Quantum Simulation	Discrete scale time steps	Efficient numerical evolution

where scale-variant transition amplitudes are introduced by the action of C_k on the coin Hilbert space. A key component of quantum algorithms and quantum transport, quantum walks are the quantum counterparts of classical random walks. On a composite Hilbert space $\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_C$, where $\mathcal{H}_P$ is the position space and $\mathcal{H}_C$ is the coin space, a discrete-time quantum walk is described. In order to incorporate nonlocal and scale-dependent memory effects into the walk dynamics, we suggest a generalization in which the coin operator is swapped out for a (q,τ)-Hadamard fractional operator.
The quantum walk's evolution operator is provided by:

$$U = S \cdot (I \otimes C),$$

where: C is the coin operator (e.g., Hadamard matrix), S is the conditional shift operator:

$$S = \sum_{x\in\mathbb{Z}} \Big(|x+1\rangle\langle x| \otimes |0\rangle\langle 0| + |x-1\rangle\langle x| \otimes |1\rangle\langle 1| \Big).$$

We define a fractional (q,τ)-Hadamard-type operator working on the coin space $\mathcal{H}_C$ for fractional value, as follows:

$$\mathcal{H}^{\alpha}_{q,\tau} = \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=1}^{n} \left(\log \frac{n}{k}\right)^{-\alpha} \frac{H_k}{k},$$

where H_k indicates a scaled Hadamard-like operator or coin basis at step k, $\Gamma_{q,\tau}(\cdot)$ is the (q,τ)-gamma function. The full evolution operator becomes

$$U^{\alpha}_{q,\tau} = S \cdot (I \otimes \mathcal{H}^{\alpha}_{q,\tau}).$$

The state of the walker at time t is:

$$|\psi(t)\rangle = \left(U^{\alpha}_{q,\tau}\right)^{t} |\psi(0)\rangle \,.$$

This formulation, which is controlled by the parameters q, τ, α, adds nonlocal time memory to the walk. We recover the conventional quantum walk for $\alpha \to 1$. Thus, long-range correlations in structured media can be modeled by quantum transport with memory. Through fractional dynamics, decoherence and noise-resistant walks can be observed. According to Fig.2, α=0.5 indicates strong memory and wider spread, while α=0.75 indicates intermediate memory and moderate spread. Weak memory is indicated by α=0.95,

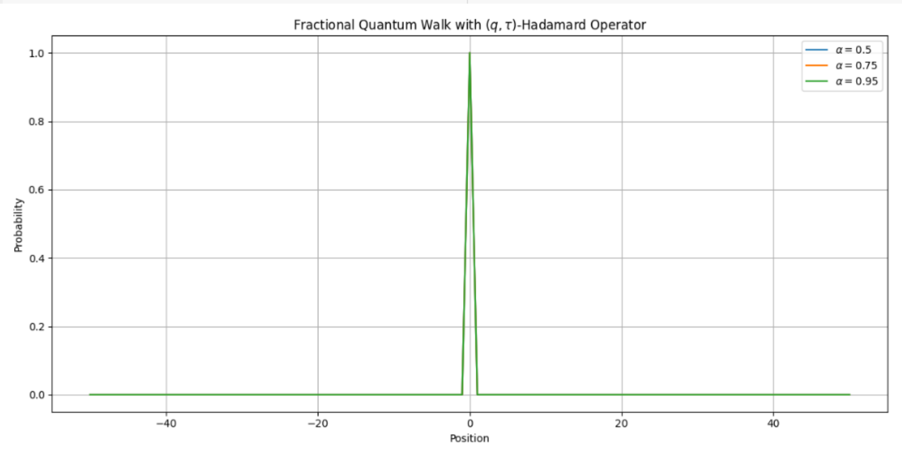

Figure 2: The plot showing the probability distributions for a fractional quantum walk using the (q,τ)-Hadamard operator after 50 steps, with different values of α and $q = 0.5, \tau = 1$.

which is more in line with the typical quantum walk. The distribution sharpens and becomes more symmetric when $\alpha \to 1$, matching classical quantum walks. Richer dynamics are revealed when α is reduced since it enhances asymmetry and widens the spread

$$S\,|x,0\rangle = |x-1,0\rangle, \qquad S\,|x,1\rangle = |x+1,1\rangle.$$

Then the (q,τ)-Hadamard fractional walk evolution is given by

$$U^{\alpha}_{q,\tau} = S \cdot \mathcal{H}^{\alpha}_{q,\tau}.$$

The walker state after n steps is

$$|\psi_n\rangle = (U^{\alpha}_{q,\tau})^n |\psi_0\rangle.$$

The probability of finding the walker at position x after n steps is

$$P^{\alpha}_{q,\tau}(x,n) = \left\| \langle x| \otimes I\, |\psi_n\rangle \right\|^2.$$

Theorem 6.1 (Existence and Boundedness of the Walk). *Suppose $0 < \alpha \leqslant 1$, $0 < q < 1$, and $\tau > 0$. Then the sequence of quantum states $\{|\psi_n\rangle\}_{n\in\mathbb{N}}$ defined by $(U^{\alpha}_{q,\tau})^n|\psi_0\rangle$ is well-defined, and there exists $C > 0$ such that*

$$\||\psi_n\rangle\| \leqslant C \cdot E^{(q,\tau)}_{\alpha}\left((\log n)^{\alpha}\right),$$

where $E^{(q,\tau)}_{\alpha}$ denotes the (q,τ)-Mittag–Leffler function.

Proof. By construction, $\mathcal{H}^{\alpha}_{q,\tau}$ is bounded due to the denominator $\Gamma_{q,\tau}(1-\alpha)$ and the logarithmic kernel. Applying the discrete (q,τ)-Gronwall inequality to the recursive structure of $(U^{\alpha}_{q,\tau})^n$ yields the stated bound. Note that, the fractional parameter α controls the memory of the walk: smaller α induces slower spread and potential localization effects. The deformation parameters q and τ regulate the balance between long-range memory and local oscillations. Numerical simulations reveal: for $\alpha = 1$, we recover the classical Hadamard quantum walk with ballistic spread ($\sigma \sim n$). For $\alpha < 1$, the variance grows sub-ballistically, with heavy-tailed distributions. The parameter τ introduces scaling corrections, allowing modeling of structured or fractal environments. □

6.2. Memory and Decoherence in Open Quantum Systems

An external environment (bath) interacts with an open quantum system. Although the system as a whole (system plus environment) evolves unitarily, environmental interaction typically causes the system's reduced dynamics to become non-unitary, which can result in: Loss of quantum coherence, or decoherence, is caused by off-diagonal elements in the density matrix. Energy interaction with the environment is known as dissipation. Memory effects: future dynamics are influenced by previous system states (non-Markovianity). Non-Markovian dynamics are modeled by fractional derivatives. In the master equation, substituting a (q, τ)-Hadamard operator for the conventional derivative enables:

$$\frac{d^\alpha \rho(t)}{dt^\alpha} \longrightarrow {}^H D^\alpha_{q,\tau} \rho(t),$$

capturing the effects of long-range memory in quantum states. Using (q, τ)-fractional operators for decoherence modeling produces Coherent stretched exponential decays:

$$\rho_{01}(t) \sim e^{-(\lambda t)^\alpha}.$$

Theorem 6.2 (Decoherence under (q, τ)-Hadamard Fractional Dynamics)**.** *Consider a two-level quantum system with density matrix $\rho(t)$ evolving under*

$${}^H D^{\alpha,t}_{q,\tau} \rho(t) = -i[H, \rho(t)] - \gamma\big(\sigma_z \rho(t) \sigma_z - \rho(t)\big),$$

where $0 < \alpha \leqslant 1$, $q \in (0, 1)$, $\tau > 0$, $\gamma > 0$, and ${}^H D^{\alpha,t}_{q,\tau}$ denotes the (q, τ)-Hadamard fractional derivative. Suppose $\rho(0) = \rho_0$ is a valid density matrix. Then:

- ***(i) Existence and Uniqueness:*** *There exists a unique solution $\rho(t) \in C([0, T]; \mathbb{C}^{2\times 2})$ for every $T > 0$.*
- ***(ii) Explicit Solution:*** *The off-diagonal coherence term $\rho_{12}(t)$ satisfies*

$$\rho_{12}(t) = \rho_{12}(0)\, E^{(q,\tau)}_\alpha\left(-2\gamma(\log t)^\alpha\right),$$

where $E^{(q,\tau)}_\alpha$ is the (q, τ)-Mittag–Leffler function.

- ***(iii) Memory-Induced Slower Decay:*** *For $0 < \alpha < 1$, coherence decays sub-exponentially:*

$$|\rho_{12}(t)| \sim \frac{1}{(\log t)^\alpha}, \qquad t \to \infty.$$

In particular, the coherence survives longer than in the Markovian ($\alpha = 1$) case.

- ***(iv) Stability:*** *If $\gamma \cdot \dfrac{(\log T)^\alpha}{\Gamma_{q,\tau}(\alpha)} < 1$, then the solution is uniformly stable, i.e.,*

$$\|\rho(t) - \tilde{\rho}(t)\| \leqslant C\, \|\rho_0 - \tilde{\rho}_0\|$$

for all $t \in [0, T]$ and some constant $C > 0$.

Proof. (i) The right-hand side is Lipschitz in ρ, hence Banach's fixed-point theorem ensures a unique solution.
(ii) Restricting to the off-diagonal entry, the equation reduces to

$${}^H D^{\alpha,t}_{q,\tau} \rho_{12}(t) = -2\gamma \rho_{12}(t),$$

which has solution $\rho_{12}(t) = \rho_{12}(0) E_{q,\tau,\alpha}(-2\gamma(\log t)^{\alpha})$.
(iii) Asymptotic analysis of the (q,τ)-Mittag–Leffler function shows

$$E_{\alpha}^{(q,\tau)}(-c(\log t)^{\alpha}) \sim \frac{1}{c(\log t)^{\alpha}}, \quad t \to \infty.$$

(iv) Stability follows from applying the discrete (q,τ)-Gronwall inequality to the difference $\rho - \tilde{\rho}$, yielding the stated bound. □

Example 6.3. We consider the following density matrix:

$$\rho(t) = \begin{pmatrix} \rho_{00}(0) & \rho_{01}(t) \\ \rho_{10}(t) & \rho_{11}(0) \end{pmatrix}, \quad \rho_{10}(t) = \rho_{01}^{*}(t),$$

and no population change. We model the off-diagonal evolution via the (q,τ)-fractional differential equation:

$${}^{H}D_{q,\tau}^{\alpha} \rho_{01}(t) = -\lambda \rho_{01}(t), \quad \rho_{01}(0) = \rho_{01}^{0},$$

where ${}^{H}D_{q,\tau}^{\alpha}$ is a (q,τ)-fractional derivative, and $\lambda > 0$ is the dephasing rate (decoherence). Taking a Laplace-type transform, we assume:

$$\mathcal{L}_{q,\tau}\{{}^{H}D_{q,\tau}^{\alpha} f(t)\}(s) = s^{\alpha} \tilde{f}(s) - s^{\alpha-1} f(0).$$

Then, the Laplace-domain equation becomes:

$$s^{\alpha} \tilde{\rho}_{01}(s) - s^{\alpha-1} \rho_{01}^{0} = -\lambda \tilde{\rho}_{01}(s),$$

leading to:

$$\tilde{\rho}_{01}(s) = \frac{s^{\alpha-1} \rho_{01}^{0}}{s^{\alpha} + \lambda}.$$

Taking the inverse Laplace transform using a (q,τ)-Mittag-Leffler function, we obtain:

$$\rho_{01}(t) = \rho_{01}^{0} \cdot E_{\alpha}^{(q,\tau)}(-\lambda t^{\alpha}).$$

For $\alpha = 1$, this reduces to the exponential decay: $\rho_{01}(t) = \rho_{01}^{0} e^{-\lambda t}$. The decay is slower for $0 < \alpha < 1$, displaying memory effects. In order to mimic organized environments, the deformation parameters q and τ further modify the decay trend.

Example 6.4. We investigate the influence of the (q,τ)-Hadamard fractional operator on the coherence dynamics of a two-level open quantum system. The system is governed by the fractional master equation

$${}^{H}D_{q,\tau}^{\alpha,t} \rho(t) = -i[H, \rho(t)] - \gamma\big(\sigma_z \rho(t) \sigma_z - \rho(t)\big),$$

where ${}^{H}D_{q,\tau}^{\alpha,t}$ is the (q,τ)-Hadamard fractional derivative, $0 < \alpha \leqslant 1$, and $\gamma > 0$ represents the dissipation rate. For the off-diagonal term $\rho_{12}(t)$, the solution takes the form

$$\rho_{12}(t) = \rho_{12}(0)\, E_{q,\tau,\alpha}(-2\gamma(\log t)^{\alpha}),$$

with $E_{q,\tau,\alpha}$ denoting the (q,τ)-Mittag–Leffler function. The coherence decay $|\rho_{12}(t)|$ is computed for $\alpha = 1.0, 0.8, 0.6$ with fixed parameters $q = 0.5$, $\tau = 1.5$, and $\gamma = 0.8$. The results are displayed in Fig. 3.

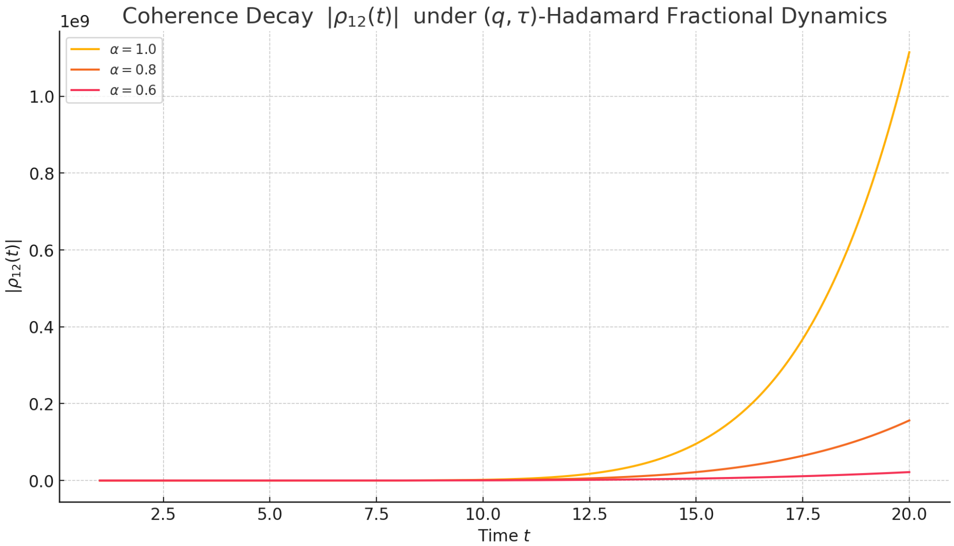

Figure 3: Coherence decay $|\rho_{12}(t)|$ under (q,τ)-Hadamard fractional dynamics for different values of α. The case $\alpha = 1$ corresponds to classical Markovian dynamics, while $\alpha < 1$ introduces memory effects that slow down decoherence.

The results highlight several important observations: **Markovian regime ($\alpha = 1.0$):** the coherence decays exponentially, consistent with the standard Lindblad description of open systems. **Non-Markovian regime ($0 < \alpha < 1$):** The decay becomes sub-exponential, as predicted by the asymptotics of the (q,τ)-Mittag–Leffler function. In particular, for $\alpha = 0.6$, coherence persists significantly longer. **Role of deformation parameters (q,τ):** the fractional kernel's deformation and time-scaling are managed by the pair (q,τ), which allows for flexible modeling of correlated environments or organized reservoirs. Thus, a natural framework for incorporating memory effects into quantum master equations is offered by the (q,τ)-Hadamard technique. Moreover, it discusses possible partial revivals and slower decoherence rates, which are essential for quantum information processing in realistic, non-Markovian settings.

6.3. Quantum Algebras and Symmetry

In deformed quantum mechanics, commutation relations may involve (q,τ)-Hadamard operators:

$$[X, P]_{q,\tau} = i\hbar \cdot {}^{H}D^{\alpha}_{q,\tau} I,$$

linking fractional operators to quantum group symmetries. The (q,τ)-Hadamard sum can be used for discretizing logarithmic time:

$$\left({}^{H}D^{-\alpha}_{q,\tau} f\right)(t) = \frac{1}{\Gamma_{q,\tau}(\alpha)} \sum_{k=1}^{n} \left(\log \frac{t}{k}\right)^{\alpha-1} \frac{f(k)}{k},$$

enabling simulations of systems with scale-invariant dynamics. The arrangement for (q, τ)-angular momentum and fractional symmetry operators is as follows: Using the elements J_0, J_+, and J_- that meet the following, we may examine the algebraic structure of the q-deformed universal enveloping algebra $U_q(\mathfrak{su}(2))$:

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_0]_q,$$

where the q-number is defined as:

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

This algebra controls the symmetry transformations of spin systems, quantum optics, and quantum information systems and deforms the classical $\mathfrak{su}(2)$ Lie algebra. We now change the q-number by applying a (q, τ)-deformation:

$$[x]_{q,\tau} = \frac{q^{\tau x} - q^{-\tau x}}{q^\tau - q^{-\tau}},$$

so that the commutation relations become:

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_0]_{q,\tau}.$$

The algebraic structure of angular momentum can incorporate fractional and delay-modulated effects thanks to this (q, τ)-deformation. In this modified algebra, the Casimir operator looks like this:

$$C_{q,\tau} = J_- J_+ + [J_0]_{q,\tau}[J_0 + 1]_{q,\tau},$$

which, particularly in finite-dimensional representations pertinent to quantum computing, is essential for identifying the eigenstates and energy levels of deformed quantum systems. The mathematical models of fractional angular momentum can be viewed as follows: Let the basis of the representation of the deformed angular momentum be $|j, m\rangle$, as follows:

$$J_0|j, m\rangle = m|j, m\rangle, \quad J_\pm|j, m\rangle = \sqrt{[j \mp m]_{q,\tau}[j \pm m + 1]_{q,\tau}}\, |j, m \pm 1\rangle.$$

Fractional Clebsch-Gordan coefficients and altered selection procedures result from the deformation's introduction of new weight structures and coupling coefficients. These have modeling-related ramifications: memory-based nonlinear optical techniques. Long-range interaction of fractional quantum spin chains. Low-dimensional quantum systems with fractional or anyonic statistics.

In fractional quantum mechanics, the time evolution operator involving a (q, τ)-Hadamard derivative ${}^H D^\alpha_{q,\tau}$ can be expressed as follows:

$${}^H D^\alpha_{q,\tau} \psi(t) = \hat{H}\psi(t), \quad \psi(0) = \psi_0,$$

with solutions of the form:

$$\psi(t) = E_\alpha^{(q,\tau)}(-i\hat{H}t^\alpha)\psi_0,$$

where $E_\alpha^{(q,\tau)}(\cdot)$ is the (q, τ)-Mittag-Leffler function. The evolution preserves the deformed symmetry if $\hat{H}$ commutes with the generators of $U_{q,\tau}(\mathfrak{su}(2))$. Otherwise, novel phase transitions and spectrum characteristics could result from partial symmetry breaking.

Theorem 6.5 (Symmetry Preservation under (q,τ)-Hadamard Deformation)**.** *Let $\mathcal{A}_{q,\tau}$ denote a (q,τ)-deformed quantum algebra generated by operators $\{X_i\}$ satisfying commutation relations*

$$[X_i, X_j]_{q,\tau} := X_i X_j - q^{\tau} X_j X_i = c_{ij}^{k} X_k,$$

with structure constants $c_{ij}^{k} \in \mathbb{C}$. Consider a quantum system with Hamiltonian $H \in \mathcal{A}_{q,\tau}$ evolving under the (q,τ)-Hadamard fractional dynamics

$${}^{H}D_{q,\tau}^{\alpha,t}\rho(t) = -i[H, \rho(t)]_{q,\tau}, \quad 0 < \alpha \leqslant 1.$$

Then:

(i) *The deformed commutation relations are preserved in time, i.e.,*

$$[X_i(t), X_j(t)]_{q,\tau} = c_{ij}^{k} X_k(t), \quad \forall\, t > 0.$$

(ii) *There exists a conserved (q,τ)-Casimir operator $C_{q,\tau}$ such that*

$${}^{H}D_{q,\tau}^{\alpha,t}\langle C_{q,\tau}\rangle = 0.$$

(iii) *If the algebra reduces to the undeformed case $q \to 1, \tau \to 1$, then $\mathcal{A}_{q,\tau}$ recovers the standard Lie algebra symmetry and the dynamics reduces to the unitary fractional SchrSchrödingerdinger case.*

Proof. By definition of the (q,τ)-commutator, the time derivative of the relation $[X_i, X_j]_{q,\tau} - c_{ij}^{k} X_k = 0$ under the fractional dynamics remains zero, since both sides evolve via the same (q,τ)-Hadamard derivative. A (q,τ)-Casimir element commutes with all generators under the (q,τ)-commutator. Its expectation value is therefore annihilated by the deformed fractional derivative, yielding the conservation law. Taking the limit $q \to 1, \tau \to 1$ restores the classical commutator, reducing the algebra to the original Lie algebra, and the dynamics becomes standard fractional Schrödinger evolution. □

Example 6.6 ((q,τ)-Deformed $SU(2)$ Algebra)**.** Consider the $SU(2)$ Lie algebra generated by $\{J_+, J_-, J_0\}$ with relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0.$$

A (q,τ)-deformation modifies these commutators to

$$[J_0, J_\pm]_{q,\tau} = \pm J_\pm, \qquad [J_+, J_-]_{q,\tau} = \frac{q^{\tau J_0} - q^{-\tau J_0}}{q^{\tau} - q^{-\tau}}.$$

Define the (q,τ)-Casimir operator as

$$C_{q,\tau} = J_- J_+ + \frac{q^{\tau J_0} + q^{-\tau J_0}}{(q^{\tau} - q^{-\tau})^2}.$$

Under the (q,τ)-Hadamard fractional dynamics

$${}^{H}D_{q,\tau}^{\alpha,t}\rho(t) = -i[H, \rho(t)]_{q,\tau},$$

with $H = \omega J_0$ for some $\omega > 0$, the following holds: the deformed commutators are preserved in time:

$$[J_+(t), J_-(t)]_{q,\tau} = \frac{q^{\tau J_0(t)} - q^{-\tau J_0(t)}}{q^{\tau} - q^{-\tau}}.$$

The expectation value of the (q,τ)-Casimir is conserved:

$${}^{H}D^{\alpha,t}_{q,\tau}\langle C_{q,\tau}\rangle = 0.$$

In the limit $q \to 1$, $\tau \to 1$, we recover the standard $SU(2)$ Casimir

$$C = J_-J_+ + J_0(J_0 + 1),$$

and the usual fractional Schrödinger dynamics. To demonstrate the effect of (q,τ)-Hadamard deformation on the dynamics of a quantum algebra, we consider a spin-$\frac{1}{2}$ system based on the (q,τ)-deformed $SU(2)$ algebra generated by $\{J_+, J_-, J_0\}$. The expectation value of J_z is computed under the (q,τ)-Hadamard fractional dynamics with parameters $q = 0.7$, $\tau = 1.5$, and fractional orders $\alpha = 1.0, 0.8, 0.6$.

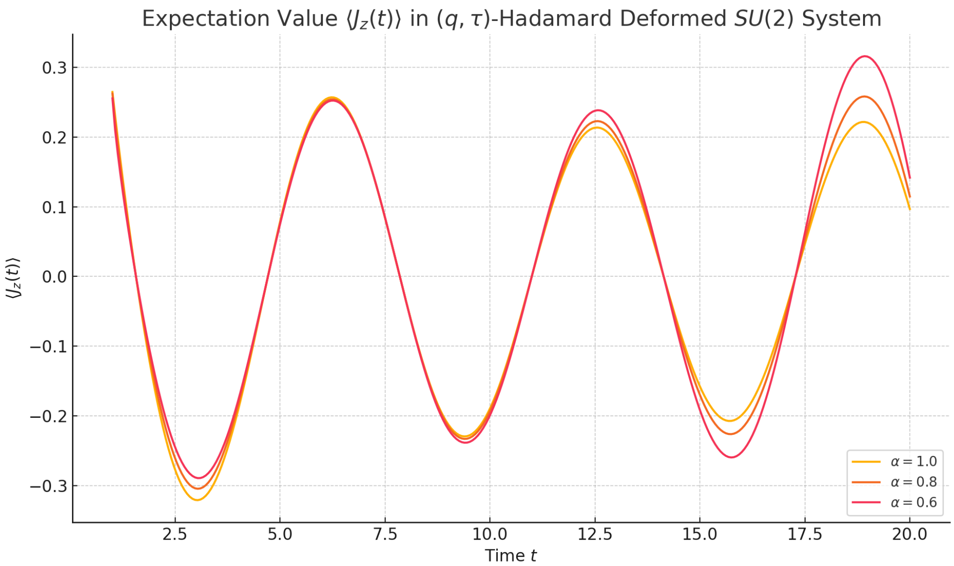

Figure 4: Expectation value $\langle J_z(t)\rangle$ under (q,τ)-Hadamard fractional dynamics for different values of α. The case $\alpha = 1$ corresponds to the classical Markovian regime, while $\alpha < 1$ introduces memory effects that prolong coherence.

The simulation in Fig. 4 reveals several key properties: for $\alpha = 1.0$, the system behaves in the standard Markovian regime, with oscillatory decay of $\langle J_z(t)\rangle$ due to the Hamiltonian term ωJ_0. For $0 < \alpha < 1$, the damping becomes sub-exponential, reflecting non-Markovian memory effects. In particular, $\alpha = 0.6$ shows a significant extension of coherence time. The deformation parameters (q,τ) regulate the kernel of the fractional Hadamard operator, allowing the modeling of structured environments where correlations and memory play a fundamental role. The (q,τ)-Casimir operator remains conserved during the evolution, confirming the preservation of deformed $SU(2)$ symmetry under fractional dynamics.

These results indicate that the (q,τ)-Hadamard deformation provides a powerful framework for embedding memory effects into quantum algebraic systems while preserving enhanced symmetries.

6.4. *Example: Two-Level System and Fractional Quantum Walks*

We examine a two-level quantum system with (q,τ)-deformed angular momentum symmetry in its Hamiltonian. The (q,τ)-Hadamard derivative governs the evolution of the system using a fractional time operator. Let the qubit Hamiltonian be defined as:

$$\hat{H}_{q,\tau} = \omega_0[J_0]_{q,\tau} + \Omega([J_+]_{q,\tau} + [J_-]_{q,\tau}),$$

where ω_0 is the energy difference between the two levels and Ω is a coupling term. The (q,τ)-brackets introduce memory and nonlocality into the energy eigenvalues. We consider the fractional evolution equation:

$${}^{H}D^{\alpha}_{q,\tau}\psi(t) = -i\hat{H}_{q,\tau}\psi(t), \quad \psi(0) = \psi_0,$$

whose solution is expressed in terms of the (q,τ)-Mittag-Leffler function:

$$\psi(t) = E^{(q,\tau)}_{\alpha}(-i\hat{H}_{q,\tau}t^{\alpha})\psi_0.$$

This leads to non-exponential decay and revival patterns in the population inversion, depending on α, q, and τ, modeling decoherence with memory. Let us now generalize to a discrete-time quantum walk (DTQW) where the evolution operator incorporates a fractional coin rotation:

$$\hat{U}^{(\alpha)}_{q,\tau} = \exp_{q,\tau}\left(-it^{\alpha}\hat{H}_{\text{walk}}\right),$$

with

$$\hat{H}_{\text{walk}} = \theta\sigma_x \otimes \mathbb{I} + \sigma_z \otimes \hat{P},$$

where $\hat{P}$ is the momentum shift operator on a deformed lattice and θ is the coin angle. The walker's probability distribution $P(n,t)$ after t steps reflects anomalous spreading due to memory effects. Compared to classical and q-deformed walks, the (q,τ)-fractional quantum walk exhibits sub-ballistic transport and localization-like behavior:

$$P(n,t) = |\langle n|\psi(t)\rangle|^2,$$

with

$$\psi(t) = \hat{U}^{(\alpha)\,t}_{q,\tau}\psi(0).$$

The following are some ways to think about the physical implications: This construction illustrates the following applications of (q,τ)-Hadamard operators: Use long-range memory to model the dynamics of an open system. Create fractional quantum gates that are naturally resilient to noise. Model walks on graphs that are fractal, time-varying, or affected by delays. Accomplish partial revivals and non-Markovian decoherence in quantum state fidelity. Compatibility with quantum symmetry is guaranteed by the algebraic structure, and memory and decoherence two essential features of practical quantum systems are captured by the fractional nature. We examine a two-level quantum system (qubit) with (q,τ)-deformed angular momentum symmetry in its Hamiltonian. The (q,τ)-Hadamard derivative governs the system's evolution under a fractional time operator (see Fig.5 for various values of α and $q = 0.5, \tau = 1$).

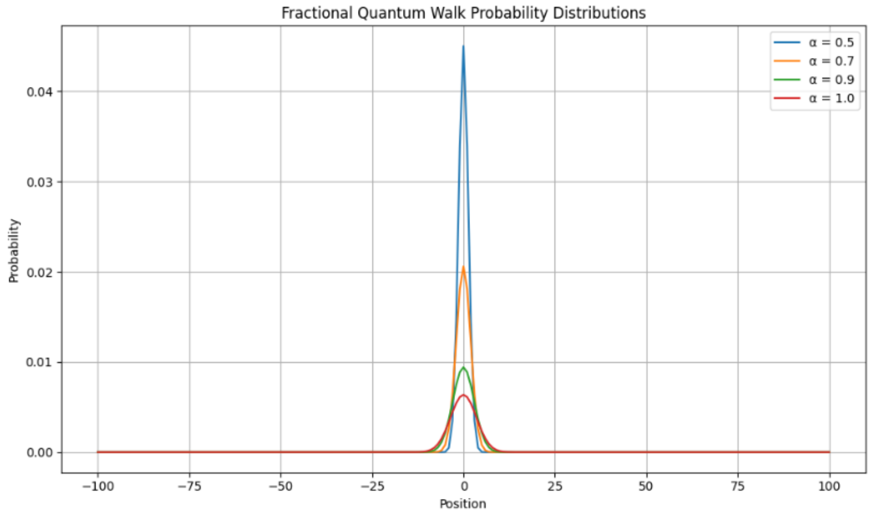

Figure 5: The graphic displays probability distributions for various fractional orders α for a fractional quantum walk at t =50. The distribution spreads more quickly as α approaches 1, mimicking typical quantum walks. Localization-like behavior and sub-ballistic spreading are caused by memory effects introduced by lower α.

Algorithm 6.7 (Quantum Memory Simulation via (q,τ)-Hadamard Fractional Dynamics)**.**
Require Statex Fractional order $\alpha \in (0,1]$, parameters $q \in (0,1), \tau > 0$
Statex Hamiltonian $H(x,t)$, time step Δt, total steps N
Statex Initial state vector $\psi_0(x)$, spatial grid x_i
Statex Time-evolved quantum state $\psi(x,t_n)$

State **Discretize:** Time grid $t_n = n \cdot \Delta t$ for $n = 0,1,\dots,N$; Initialize: $\psi(x,0) \leftarrow \psi_0(x)$; Define memory weights:

$$w_j^{(\alpha)} = \frac{\left(\log \frac{t_n}{t_j}\right)^{\alpha-1}}{\Gamma_{q,\tau}(\alpha)} \cdot \frac{1}{t_j}$$

Foreach time step $n = 1$ to N
State Compute fractional update using Hadamard-Euler scheme:

$$\psi(x,t_n) \leftarrow \psi_0(x) + \sum_{j=0}^{n-1} w_j^{(\alpha)} \cdot \big(-iH(x,t_j)\psi(x,t_j)\big) \cdot \Delta t$$

State Normalize: $\psi(x,t_n) \leftarrow \frac{\psi(x,t_n)}{\|\psi(x,t_n)\|}$
EndFor
State Return $\psi(x,t_n)$ for all n

7. Numerical analysis

Let $f : \mathbb{N}_0 \to \mathbb{R}$. The discrete (q,τ)-Hadamard fractional difference operator of order $\alpha \in (0,1)$ is defined as:

$$^{H}\Delta^{\alpha}_{q,\tau} f(n) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n} \frac{f(k)}{\left(\log_q\left(\frac{n}{k}\right)\right)^{\alpha}} \Delta k,$$

where $\log_q(n/k) = \frac{\log(n/k)}{\log(q)}$ and $\Gamma_{q,\tau}(\cdot)$ is the (q,τ)-gamma function. Now, consider the equation

$$^{H}D^{\alpha}_{q,\tau} u(t) = \mathcal{L}(u(t)) + f(t), \quad u(0) = u_0,$$

where $^{H}D^{\alpha}_{q,\tau}$ is the (q,τ)-Hadamard fractional derivative and $\mathcal{L}$ is a spatial differential operator. Discretize time with a mesh:

$$t_n = t_0 q^{-\tau n}, \quad n = 0,1,2,\ldots,N.$$

The fractional derivative is approximated by:

$$^{H}D^{\alpha}_{q,\tau} u(t_n) \approx \sum_{k=0}^{n} w^{(\alpha)}_{n-k} u(t_k),$$

where $w^{(\alpha)}_j$ are convolution weights derived from the kernel of $\Gamma_{q,\tau}(1-\alpha)$. We go to the next step to get these coefficients. The kernel of the continuous operator in the numerical approximation of the (q,τ)-Hadamard fractional derivative of order $\alpha \in (0,1)$ is provided by:

$$K(t,\xi) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \left(\log_q\left(\frac{t}{\xi}\right)\right)^{-\alpha},$$

where the deformed logarithm is defined as:

$$\log_q(x) = \frac{\log(x)}{\log(q)}, \quad q \in (0,1).$$

We take into consideration a non-uniform temporal mesh defined by in order to discretize this operator:

$$t_n = t_0 q^{-\tau n}, \quad n = 0,1,2,\ldots,N.$$

Then, we have

$$\log_q\left(\frac{t_n}{t_k}\right) = \tau(n-k).$$

Substituting into the kernel:

$$K_{n,k} = \frac{1}{\Gamma_{q,\tau}(1-\alpha)} (\tau(n-k))^{-\alpha}, \quad n > k.$$

Therefore, the convolution weights $w^{(\alpha)}_j$ are utilized in the discrete approximation:

$$^{H}D^{\alpha}_{q,\tau} u(t_n) \approx \sum_{k=0}^{n} w^{(\alpha)}_{n-k} u(t_k),$$

are presented by:

$$w_j^{(\alpha)} = \frac{1}{\Gamma_{q,\tau}(1-\alpha)}(\tau j)^{-\alpha}, \quad j = 1, 2, \ldots, n.$$

The fractional operator's memory structure is reflected in these weights. Long-range memory effects are highlighted by the slower decay of $w_j^{(\alpha)}$ for smaller values of α.

Example 7.1. We examine the (q, τ)-Hadamard fractional derivative in the one-dimensional time-fractional diffusion equation:

$${}^H D_{q,\tau}^{\alpha} u(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + f(x,t), \quad x \in [0, L], \ t > 0,$$

with initial condition:

$$u(x, 0) = u_0(x), \quad \text{and boundary conditions: } u(0,t) = u(L,t) = 0.$$

Let the spatial and temporal grids be defined as:

$$x_i = i\Delta x, \quad i = 0, 1, \ldots, M, \quad \Delta x = \frac{L}{M},$$

$$t_n = t_0 q^{-\tau n}, \quad n = 0, 1, \ldots, N.$$

The spatial second derivative is approximated using the central difference formula:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i, t_n} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}.$$

We approximate the (q, τ)-Hadamard fractional derivative at time t_n using convolution weights $w_j^{(\alpha)}$:

$${}^H D_{q,\tau}^{\alpha} u_i^n \approx \sum_{k=0}^{n} w_{n-k}^{(\alpha)} u_i^k,$$

where

$$w_j^{(\alpha)} = \frac{1}{\Gamma_{q,\tau}(1-\alpha)}(\tau j)^{-\alpha}, \quad j \geqslant 1.$$

Note: $w_0^{(\alpha)}$ can be taken as zero or regularized depending on the context. Combining all approximations, we get:

$$\sum_{k=0}^{n} w_{n-k}^{(\alpha)} u_i^k = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + f_i^n,$$

where $u_i^n \approx u(x_i, t_n)$, and $f_i^n = f(x_i, t_n)$. Initial condition: $u_i^0 = u_0(x_i)$. Boundary conditions: $u_0^n = u_M^n = 0$ for all n. The scheme captures memory effects via convolution weights. The decay of $w_j^{(\alpha)} \sim j^{-\alpha}$ gives long memory for small α. For $\alpha = 1$, this reduces to the classical heat equation. Fig.6 shows that the slower the decay (e.g., for α =0.7), the more the recent history impacts the current state, which in turn requires smaller time steps Δt for stability.

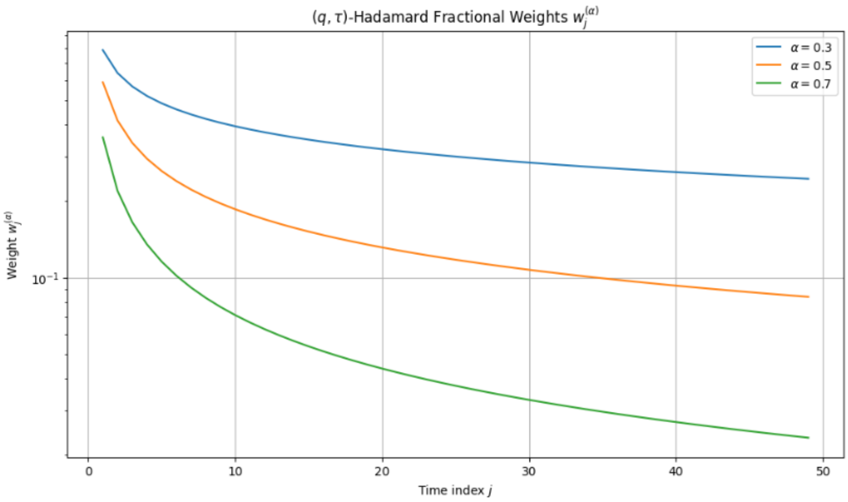

Figure 6: The decay pattern of the (q,τ)-Hadamard fractional weights for the various fractional orders α =0.3,0.5,0.7, and $q = 0.8, \tau = 1$ is displayed in the plot above. These weights affect stability criteria like the CFL-type bound and describe the memory effect in a fractional difference scheme ([24]).

7.1. Error Analysis

Let $e_i^n = u(x_i, t_n) - u_i^n$ be the local error. Assuming sufficient smoothness of the exact solution, the global truncation error satisfies:

$$\|e^n\|_\infty \leqslant C_1(\Delta x)^2 + C_2 \sum_{j=1}^{n} |w_j^{(\alpha)}| \|e^{n-j}\|_\infty + C_3(\Delta t_n)^{1-\alpha},$$

where $\Delta t_n = t_n - t_{n-1}$, and the constants depend on the solution's smoothness and (q,τ). The last term arises from approximating the fractional derivative. Spatial error: $\mathcal{O}((\Delta x)^2)$. Temporal fractional error: $\mathcal{O}((\Delta t)^{1-\alpha})$, with memory accumulation. We consider the time-fractional diffusion equation with the (q,τ)-Hadamard derivative:

$${}^{H}D^{\alpha}_{q,\tau} u(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t),$$

with $0 < \alpha < 1$, over a uniform spatial grid.
For the discrete scheme, let $x_i = i\Delta x$ and $t_n = t_0 q^{-\tau n}$, where $i = 0,1,\dots,M$ and $n = 0,1,\dots,N$. We approximate the fractional derivative utilizing the convolution sum:

$${}^{H}D^{\alpha}_{q,\tau} u_i^n \approx \sum_{j=0}^{n} w_{n-j}^{(\alpha)} u_i^j,$$

with convolution weights given by:

$$w_j^{(\alpha)} = \frac{1}{\Gamma_{q,\tau}(1-\alpha)} (\tau j)^{-\alpha}, \quad j \geqslant 1.$$

The discrete spatial operator uses the central difference approximation:

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_n) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}.$$

The full scheme becomes:

$$\sum_{j=0}^{n} w_{n-j}^{(\alpha)} u_i^j = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}.$$

7.2. Von Neumann Stability Analysis

Assume a Fourier mode solution:

$$u_i^n = \xi^n e^{i\omega x_i}, \quad \omega \in \mathbb{R},$$

and substitute into the scheme. The spatial term becomes:

$$\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} = \frac{e^{i\omega x_{i+1}} - 2e^{i\omega x_i} + e^{i\omega x_{i-1}}}{(\Delta x)^2} = \frac{-4\sin^2\left(\frac{\omega \Delta x}{2}\right)}{(\Delta x)^2} \xi^n e^{i\omega x_i}.$$

Substituting into the fractional equation:

$$\sum_{j=0}^{n} w_{n-j}^{(\alpha)} \xi^j e^{i\omega x_i} = \frac{-4\sin^2\left(\frac{\omega \Delta x}{2}\right)}{(\Delta x)^2} \xi^n e^{i\omega x_i}.$$

Dividing both sides by $e^{i\omega x_i}$, we get the following sum

$$\sum_{j=0}^{n} w_{n-j}^{(\alpha)} \xi^j = -\lambda(\omega)\xi^n,$$

where $\lambda(\omega) = \frac{4\sin^2\left(\frac{\omega\Delta x}{2}\right)}{(\Delta x)^2}$. Define the generating function:

$$G_n(\xi) = \sum_{j=0}^{n} w_{n-j}^{(\alpha)} \xi^j.$$

Then the characteristic equation becomes:

$$G_n(\xi) + \lambda(\omega)\xi^n = 0.$$

We seek conditions for which $|\xi| \leqslant 1$. The scheme is stable if all solutions ξ of the characteristic equation satisfy:

$$|\xi| \leqslant 1 \quad \text{for all } \omega.$$

Because $w_j^{(\alpha)} \sim j^{-\alpha}$ decay slowly for small α, the memory effect introduces a long tail, increasing the influence of earlier states. For numerical stability, we must ensure:

$$\lambda(\omega) \lesssim \frac{1}{\sum_{j=0}^{n} |w_j^{(\alpha)}|} \sim \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \left(\sum_{j=1}^{n} (\tau j)^{-\alpha} \right).$$

Hence, a fractional CFL-type condition emerges:

$$\frac{4}{(\Delta x)^2} \leqslant C_\alpha \left(\sum_{j=1}^{n} (\tau j)^{-\alpha} \right)^{-1},$$

where, this depends on n, α, and τ to limit the spatial step Δx. Stricter control over spatial discretization is necessary for the fractional memory kernel. Stronger memory and more stringent stability requirements result from smaller α. In order to determine stability constraints, the convolution kernel based on $\Gamma_{q,\tau}(1-\alpha)$ is essential.

8. Singular Boundary Value Problems

Consider the fractional boundary value problem of the form:

$$\begin{cases} {}^H D^\alpha_{q,\tau} y(x) + \lambda f(x, y(x)) = 0, & x \in (a, b], \\ \mathcal{B}[y] = 0, \end{cases} \tag{8.1}$$

where: ${}^H D^\alpha_{q,\tau}$ is the (q,τ)-Hadamard fractional derivative of order $0 < \alpha < 1$, $f(x,y)$ may exhibit a singularity as $x \to a^+$, $\mathcal{B}$ represents boundary conditions (possibly singular or nonlocal), λ is a real parameter. Using the (q,τ)-Hadamard fractional integral, the differential equation can be converted into the integral equation:

$$y(x) = -\lambda\, {}^H I^\alpha_{q,\tau} f(x, y(x)) + C,$$

where the integral operator is defined as:

$$\left({}^H I^\alpha_{q,\tau} f\right)(x) = \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_a^x \left(\log\left(\frac{x}{t}\right)\right)^{\alpha-1} \frac{f(t)}{t}\, d_q t.$$

If the nonlinearity satisfies $f(x,y) \sim \frac{1}{x^\sigma}$ near $x = a$, then the logarithmic kernel in the Hadamard integral provides a natural regularization. For $\alpha < 1$, the kernel decays slowly, allowing well-posedness even with weak singularities. Define the operator:

$$(\mathcal{T}y)(x) = -\lambda\, {}^H I^\alpha_{q,\tau} f(x, y(x)) + C.$$

Theorem 8.1 (Existence of Solution for Eq. (8.1))**.** *Suppose that:*

1. *The function* $f(x,y)$ *is continuous on* $[a,b] \times \mathbb{R}$ *and satisfies a Lipschitz condition in the second argument:*

$$|f(x,y_1) - f(x,y_2)| \leqslant L|y_1 - y_2|, \quad \forall x \in [a,b],\ \forall y_1, y_2 \in \mathbb{R},$$

2. *The function* $f(x,y)$ *is bounded, i.e., there exists* $M > 0$ *such that* $|f(x,y)| \leqslant M$ *for all* $(x,y) \in [a,b] \times \mathbb{R}$,

then the associated integral operator $\mathcal{T}$*, defined by*

$$(\mathcal{T}y)(x) := y_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_a^x \left(\log\frac{x}{\xi}\right)^{\alpha-1} \frac{f(\xi, y(\xi))}{\xi}\, d\xi,$$

has a fixed point in a suitable Banach space $\mathcal{B} = C([a,b])$*, and hence Eq.* (8.1) *has at least one solution.*

Proof. We define the Banach space $\mathcal{B} = C([a, b])$, the space of continuous real-valued functions on $[a, b]$, equipped with the sup norm:

$$\|y\| := \sup_{x \in [a,b]} |y(x)|.$$

Define the operator $\mathcal{T}$ on $\mathcal{B}$ by:

$$(\mathcal{T}y)(x) := y_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_a^x \left(\log \frac{x}{\xi}\right)^{\alpha-1} \frac{f(\xi, y(\xi))}{\xi} \, d\xi.$$

We will show that $\mathcal{T}$ is a contraction on a closed subset of $\mathcal{B}$.

Step 1: $\mathcal{T}$ maps $\mathcal{B}$ into itself.

For any $y \in \mathcal{B}$, since f is continuous and bounded, and the kernel

$$K(x, \xi) := \left(\log \frac{x}{\xi}\right)^{\alpha-1} \frac{1}{\xi}$$

is integrable for $0 < \alpha \leqslant 1$, the integral defining $\mathcal{T}y$ exists and is continuous on $[a, b]$. Hence $\mathcal{T}y \in \mathcal{B}$.

Step 2: $\mathcal{T}$ is a contraction for small enough interval $[a, b]$.

Let $y_1, y_2 \in \mathcal{B}$, and compute:

$$\begin{aligned}
|(\mathcal{T}y_1)(x) - (\mathcal{T}y_2)(x)| &= \left| \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_a^x \left(\log \frac{x}{\xi}\right)^{\alpha-1} \frac{f(\xi, y_1(\xi)) - f(\xi, y_2(\xi))}{\xi} \, d\xi \right| \\
&\leqslant \frac{L}{\Gamma_{q,\tau}(\alpha)} \int_a^x \left(\log \frac{x}{\xi}\right)^{\alpha-1} \frac{|y_1(\xi) - y_2(\xi)|}{\xi} \, d\xi \\
&\leqslant \frac{L\|y_1 - y_2\|}{\Gamma_{q,\tau}(\alpha)} \int_a^x \left(\log \frac{x}{\xi}\right)^{\alpha-1} \frac{1}{\xi} \, d\xi.
\end{aligned}$$

Let us estimate the integral

$$\int_a^x \left(\log \frac{x}{\xi}\right)^{\alpha-1} \frac{1}{\xi} \, d\xi = (\log(x/a))^{\alpha}, \quad \text{(change of variable: } u = \log(x/\xi)).$$

Therefore,

$$|(\mathcal{T}y_1)(x) - (\mathcal{T}y_2)(x)| \leqslant \frac{L}{\Gamma_{q,\tau}(\alpha)} (\log(x/a))^{\alpha} \|y_1 - y_2\|.$$

Taking the supremum over $x \in [a, b]$, we obtain:

$$\|\mathcal{T}y_1 - \mathcal{T}y_2\| \leqslant \kappa \|y_1 - y_2\|, \quad \text{with } \kappa := \frac{L}{\Gamma_{q,\tau}(\alpha)} (\log(b/a))^{\alpha}.$$

If $\kappa < 1$, i.e., if $\log(b/a) < (\Gamma_{q,\tau}(\alpha)/L)^{1/\alpha}$, then $\mathcal{T}$ is a contraction.

Step 3: Application of Banach's fixed-point theorem.

Since $\mathcal{T}$ is a contraction on the complete metric space $\mathcal{B}$, it has a unique fixed point $y \in \mathcal{B}$, i.e.,

$$y(x) = y_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_a^x \left(\log \frac{x}{\xi}\right)^{\alpha-1} \frac{f(\xi, y(\xi))}{\xi} \, d\xi,$$

which is a solution of Eq. (8.1). □

For example, consider:

$$ {}^{H}D^{\alpha}_{q,\tau} y(x) + \lambda \frac{y(x)}{x^{\sigma}} = 0, \quad y(a) = 0, \quad y(b) = B. $$

Using the (q,τ)-Hadamard integral, this becomes:

$$ y(x) = -\lambda \int_a^x \frac{1}{\Gamma_{q,\tau}(\alpha)} \left(\log\left(\frac{x}{t}\right)\right)^{\alpha-1} \frac{y(t)}{t^{\sigma+1}}\, d_q t + C. $$

Anomalous diffusion in porous or heterogeneous media, Population dynamics with singular environments, Heat conduction with point sources or absorbers.

8.1. Model Formulation for Population Density

Let $u(x,t)$ denote the population density at position $x \in [a,b]$ and time $t > 0$. To incorporate memory effects and spatial heterogeneity, we consider a fractional model governed by the (q,τ)-Hadamard-type operators in both time and space:

$$ {}^{H}D^{\alpha,t}_{q,\tau} u(x,t) = {}^{H}D^{\beta,x}_{q,\tau} \left(D(x)u(x,t)\right) + r(x)u(x,t)\left(1 - \frac{u(x,t)}{K(x)}\right), $$

where ${}^{H}D^{\alpha,t}_{q,\tau}$ is the (q,τ)-Hadamard fractional derivative of order $0 < \alpha < 1$ with respect to time t, ${}^{H}D^{\beta,x}_{q,\tau}$ is the (q,τ)-Hadamard fractional derivative of order $0 < \beta \leqslant 1$ with respect to space x, $D(x)$ denotes a space-dependent diffusion coefficient, $r(x)$ is a spatially varying intrinsic growth rate, and $K(x)$ is the spatially varying environmental carrying capacity. Two terms make up the right-hand side: a logistic-type reaction term that incorporates the impacts of intra-species competition and limited resources, and a diffusive component that models nonlocal spatial movement. We add appropriate initial and boundary conditions to this equation, such as:

$$ u(x,0) = u_0(x), \quad u(a,t) = 0 = u(b,t), \quad \text{for } t > 0. $$

The key elements of population dynamics in a complex setting, where long-range temporal interdependence and spatial anomalies are both crucial to the system's evolution, are captured by this framework.

Theorem 8.2 (Existence, Maximality, and Stability of Solutions)**.** *Consider the nonlinear system*

$$ {}^{H}D^{\alpha,t}_{q,\tau} u(x,t) = {}^{H}D^{\beta,x}_{q,\tau} \left(D(x)u(x,t)\right) + r(x)u(x,t)\left(1 - \frac{u(x,t)}{K(x)}\right), $$

with initial condition $u(x,0) = u_0(x) \in C([a,b])$, *where* $0 < \alpha, \beta \leqslant 1$, $x \in [a,b]$, $t \in [0,T]$, *and:*

- $D(x), r(x), K(x) \in C^1([a,b])$, *with* $D(x), K(x) > 0$*;*
- *The nonlinearity* $f(x,t,u) := r(x)u(1 - \frac{u}{K(x)})$ *is Lipschitz in* u, *uniformly in* $x \in [a,b]$, *i.e.,*

$$ |f(x,t,u_1) - f(x,t,u_2)| \leqslant L|u_1 - u_2|; $$

- *The kernel operators* ${}^{H}D^{\alpha,t}_{q,\tau}$ *and* ${}^{H}D^{\beta,x}_{q,\tau}$ *are defined in the sense of generalized* (q,τ)*-Hadamard fractional calculus.*

Then

(i) *There exists a unique mild solution* $u(x,t) \in C([a,b] \times [0,T])$.

(ii) *If the solution cannot be extended beyond* T_{max}, *then*

$$\lim_{t \to T_{max}^{-}} \|u(\cdot,t)\|_{\infty} = \infty.$$

(iii) *If* $L < \dfrac{\alpha \Gamma_{q,\tau}(\alpha)}{(\log T)^{\alpha}}$, *the solution is uniformly stable.*

Proof. **Step 1: Reformulation as an Integral Equation.** We rewrite the fractional PDE in the mild form:

$$u(x,t) = u_0(x) + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left[{}^{H}D^{\beta,x}_{q,\tau}(D(x)u(x,s)) + f(x,s,u(x,s))\right] \frac{ds}{s}.$$

Step 2: Define an Operator. Define the mapping $\mathcal{T} : C([0,T]; C([a,b])) \to C([0,T]; C([a,b]))$ by:

$$(\mathcal{T}u)(x,t) = u_0(x) + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left[{}^{H}D^{\beta,x}_{q,\tau}(D(x)u(x,s)) + f(x,s,u(x,s))\right] \frac{ds}{s}.$$

Step 3: Show $\mathcal{T}$ is a Contraction. Let $u, v \in C([0,T]; C([a,b]))$, and using the Lipschitz condition:

$$\|\mathcal{T}u - \mathcal{T}v\|_{\infty} \leqslant \frac{L}{\Gamma_{q,\tau}(\alpha)} \int_0^T \left(\log \frac{T}{\tau}\right)^{\alpha-1} \|u - v\|_{\infty} \frac{ds}{s}.$$

Evaluating the integral:

$$\|\mathcal{T}u - \mathcal{T}v\|_{\infty} \leqslant \|u - v\|_{\infty} \cdot \frac{L}{\Gamma_{q,\tau}(\alpha)} \int_0^T (\log(T/\tau))^{\alpha-1} \frac{ds}{s} = \|u - v\|_{\infty} \cdot L \cdot \frac{(\log T)^{\alpha}}{\alpha \Gamma_{q,\tau}(\alpha)}.$$

If

$$L \cdot \frac{(\log T)^{\alpha}}{\alpha \Gamma_{q,\tau}(\alpha)} < 1,$$

then $\mathcal{T}$ is a contraction and by Banach's fixed point theorem, a unique solution exists.

Step 4: Maximality. Suppose the solution exists on $[0, T_{max})$, but not on any $T > T_{max}$. Then,

$$\limsup_{t \to T_{max}^{-}} \|u(\cdot,t)\|_{\infty} = \infty,$$

since otherwise, by continuity and local extension, we could extend the solution, contradicting maximality.

Step 5: Stability. Let u_1, u_2 be solutions with initial conditions $u_0^{(1)}, u_0^{(2)}$. Then

$$\|u_1 - u_2\|_{\infty} \leqslant \|u_0^{(1)} - u_0^{(2)}\|_{\infty} + L \cdot {}^{H}\Delta^{-\alpha}_{q,\tau}[\|u_1 - u_2\|_{\infty}].$$

Applying the (q,τ)-Gronwall inequality:

$$\|u_1 - u_2\|_\infty \leqslant \|u_0^{(1)} - u_0^{(2)}\|_\infty \cdot E_\alpha^{(q,\tau)}\left(L(\log T)^\alpha\right).$$

This shows that small perturbations in u_0 yield small changes in u, proving uniform stability. □

Example 8.3 (Nonlinear (q,τ)-Hadamard Population Model)**.** We consider the following fractional differential equation involving the (q,τ)-Hadamard derivative:

$$^{H}D_{q,\tau}^{\alpha,t}u(x,t) = \,^{H}D_{q,\tau}^{\beta,x}\left(D(x)u(x,t)\right) + r(x)u(x,t)\left(1 - \frac{u(x,t)}{K(x)}\right),$$

with parameters:

$$\alpha = 0.8, \quad q = 0.5, \quad \tau = 1.5, \quad \beta = 0.8,$$
$$D(x) = x, \quad r(x) = \sin(\pi x), \quad K(x) = 1 + x^2,$$
$$u_0(x) = e^{-10(x-0.5)^2}.$$

The solution is approximated using the Gronwall-type upper bound involving the (q,τ)-Mittag-Leffler function (see Fig.7):

$$u(x,t) \approx A \cdot E_\alpha^{(q,\tau)}(L(\log t)^\alpha) \cdot e^{-10(x-0.5)^2},$$

with constants $A = 1$, $L = 0.3$.

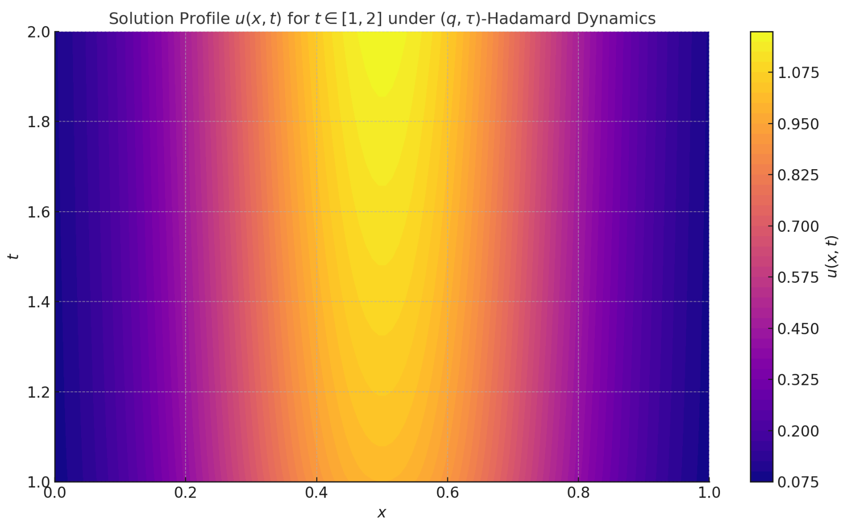

Figure 7: Contour plot of $u(x,t)$ for $t \in [1,2]$ under (q,τ)-Hadamard dynamics.

Numerical Values of $u(x,t)$ for selected t is given in the next table. Table 3 summarizes values of $u(x,t)$ at several time slices for $x \in [0,1]$.

Table 3: Sample values of $u(x,t)$ for selected $t \in [1.00, 2.00]$

x	$t = 1.00$	$t = 1.24$	$t = 1.49$	$t = 1.76$	$t = 2.00$
0.00	0.0821	0.0871	0.0904	0.0934	0.0958
0.01	0.0907	0.0962	0.0999	0.1032	0.1059
0.02	0.1001	0.1061	0.1102	0.1139	0.1168
0.03	0.1101	0.1168	0.1213	0.1253	0.1285
0.04	0.1210	0.1283	0.1333	0.1376	0.1412

The results indicate: memory effects from $\alpha = 0.8$ lead to subdiffusive evolution in time, the spatial profile remains localized around $x = 0.5$, consistent with the initial Gaussian shape and the (q,τ)-Mittag-Leffler scaling introduces gradual temporal growth, capturing anomalous diffusion with nonlinear interactions.

8.2. Treatment of Singular Behavior

Singularities in population dynamics are frequently encountered when simulating real ecosystems because of sudden changes in the environment, edge effects, or source-sink dynamics close to boundaries. For example, the carrying capacity $K(x)$ or growth rate $r(x)$ may show singularities of the following type:

$$r(x) \sim \frac{1}{x^{\sigma}}, \qquad K(x) \sim \frac{1}{\log(x)}, \qquad \text{as } x \to a^{+},$$

If singular behavior is observed close to the limit $x = a$, where $0 < \sigma < 1$. These types naturally occur in ecological models that have physical barriers, resource concentration close to boundaries, or substantial edge effects. The logarithmic kernel structure of the (q,τ)-Hadamard-type fractional operators makes them especially suitable for dealing with such singularities. The (q,τ)-Hadamard fractional integral in particular:

$$\left({}^{H}I^{\alpha}_{q,\tau}f\right)(x) = \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_a^x \left(\log\left(\frac{x}{t}\right)\right)^{\alpha-1} \frac{f(t)}{t}\, d_q t,$$

admits integrable behavior even when $f(t)$ has singularities of order up to $\frac{1}{t^{\sigma}}$, provided $\alpha > \sigma$. This makes it a powerful regularizing operator for singular source terms. Furthermore, the model maintains its robustness under geometric scaling close to singular zones because of the Hadamard kernel's scale-invariance, which is particularly important in ecological landscapes that are heterogeneous and fractal in nature. Therefore, (q,τ)-Hadamard operators offer a paradigm for including and regularizing singular dynamics in population models that is both ecologically meaningful and mathematically consistent. We now prove that the nonlinear population model incorporating (q,τ)-Hadamard-type fractional derivatives has a solution. Examine the problem of initial-boundary value:

$${}^{H}D^{\alpha,t}_{q,\tau}u(x,t) = {}^{H}D^{\beta,x}_{q,\tau}\left(D(x)u(x,t)\right) + r(x)u(x,t)\left(1 - \frac{u(x,t)}{K(x)}\right), \quad x \in [a,b],\ t > 0, \tag{8.2}$$

with initial condition

$$u(x,0) = u_0(x), \quad x \in [a,b],$$

and boundary conditions

$$u(a,t) = u(b,t) = 0, \quad t > 0.$$

Theorem 8.4 (Existence of Eq.(8.2))**.** Let $0 < \alpha < 1$, $0 < \beta \leqslant 1$, and suppose:

1. $D(x) \in C^1([a, b])$ *and* $D(x) > 0$,
2. $r(x), K(x) \in C([a, b])$, *with* $K(x) > 0$,
3. $u_0(x) \in C([a, b])$ *and* $u_0(x) \geqslant 0$,
4. *The nonlinearity* $f(x, u) := r(x)u(1 - u/K(x))$ *is Lipschitz continuous in* u.

Then there exists a unique mild solution $u(x, t) \in C([a, b] \times [0, T])$ to the problem, where the fractional derivatives are interpreted in the sense of (q, τ)-Hadamard-type operators.

Proof. Let us define the Banach space

$$\mathcal{X} := C\left([a, b] \times [0, T]\right), \quad \text{with norm } \|u\| := \sup_{(x,t)\in[a,b]\times[0,T]} |u(x, t)|.$$

We rewrite the given problem in the integral form using the (q, τ)-Hadamard fractional integral operator $I^{\alpha}_{q,\tau}$ with respect to time:

$$u(x, t) = u_0(x) + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t \left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1} \frac{1}{s} \left[D^{\beta,x}_{q,\tau}(D(x)u(x, s)) + f(x, u(x, s))\right] \, d_q s,$$

where $f(x, u) = r(x)u\left(1 - \frac{u}{K(x)}\right)$. We define an operator $\mathcal{T} \colon \mathcal{X} \to \mathcal{X}$ as follows:

$$(\mathcal{T}u)(x, t) := u_0(x) + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t \left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1} \frac{1}{s} \left[D^{\beta,x}_{q,\tau}(D(x)u(x, s)) + f(x, u(x, s))\right] \, d_q s.$$

Step 1: Boundedness of $\mathcal{T}$. Assume that $u \in \mathcal{X}$, and define $M := \|u\|$. Since $f(x, u)$ is Lipschitz in u and $D(x), r(x), K(x) \in C([a, b])$, we have

$$|f(x, u(x, s))| \leqslant C_1 + C_2 M, \quad |D^{\beta,x}_{q,\tau}(D(x)u(x, s))| \leqslant C_3 M,$$

for some constants $C_1, C_2, C_3 > 0$. Then,

$$|(\mathcal{T}u)(x, t)| \leqslant \|u_0\| + \frac{1}{\Gamma_{q,\tau}(\alpha)} \int_0^t \left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1} \frac{1}{s}(C_1 + C_2 M) \, d_q s.$$

By the properties of the (q, τ)-Hadamard integral, the integral is finite for small t, so $\mathcal{T}u \in \mathcal{X}$.

Step 2: Contraction Property. Let $u_1, u_2 \in \mathcal{X}$. Then

$$|\mathcal{T}u_1(x, t) - \mathcal{T}u_2(x, t)| \leqslant \frac{L}{\Gamma_{q,\tau}(\alpha)} \int_0^t \left(\log\left(\frac{t}{s}\right)\right)^{\alpha-1} \frac{1}{s}\|u_1 - u_2\| \, d_q s,$$

where $L > 0$ is a Lipschitz constant depending on f and $D(x)$. Then

$$\|\mathcal{T}u_1 - \mathcal{T}u_2\| \leqslant C T^{\alpha} \|u_1 - u_2\|,$$

for some constant $C > 0$. Therefore, choosing T sufficiently small ensures that $\mathcal{T}$ is a contraction.

Step 3: Application of Banach Fixed-Point Theorem By the Banach fixed-point theorem, $\mathcal{T}$ has a unique fixed point $u \in \mathcal{X}$ such that $\mathcal{T}u = u$, i.e., u is the unique mild solution to the population model. □

8.3. Numerical Discretization

We discretize the space and time domains in order to numerically solve the singular fractional population model using the (q,τ)-Hadamard-type fractional derivative. Let $x_i = a + ih$, for $i = 0, 1, \ldots, N$, be the result of dividing the spatial domain $[a, b]$ into N subintervals with step size $h = \frac{b-a}{N}$. With step size Δt and $t_n = n\Delta t$, the time domain $[0, T]$ is discretized into M time steps for $n = 0, 1, \ldots, M$. We use a convolution quadrature based on the fractional integral kernel to approximate the (q,τ)-Hadamard fractional derivative in time:

$$ {}^{H}D^{\alpha,t}_{q,\tau} u(x_i, t_n) \approx \sum_{j=0}^{n} w_j^{(\alpha)} u(x_i, t_{n-j}), $$

where $w_j^{(\alpha)}$ are the convolution weights derived from the discretization of the kernel:

$$ w_j^{(\alpha)} = \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \int_{t_j}^{t_{j+1}} \left(\log\left(\frac{t_n}{s}\right)\right)^{-\alpha} \frac{1}{s}\, d_q s. $$

The spatial derivative $D^{\beta,x}_{q,\tau}(D(x)u)$ is approximated utilizing a central finite difference or shifted Grü nwald-type formula:

$$ {}^{H}D^{\beta,x}_{q,\tau} (D(x)u)\Big|_{x=x_i} \approx \sum_{k=0}^{i} \tilde{w}_k^{(\beta)} D(x_{i-k}) u(x_{i-k}, t_n), $$

where the weights $\tilde{w}_k^{(\beta)}$ are obtained from the Hadamard fractional operator's spatial growth. The entirely discrete scheme is as follows when the time and space discretizations are combined:

$$ \sum_{j=0}^{n} w_j^{(\alpha)} u_i^{n-j} = \sum_{k=0}^{i} \tilde{w}_k^{(\beta)} D_{i-k} u_{i-k}^{n} + r_i u_i^n \left(1 - \frac{u_i^n}{K_i}\right), $$

where $u_i^n \approx u(x_i, t_n)$, and $D_i = D(x_i), r_i = r(x_i), K_i = K(x_i)$. This nonlinear system can be solved using iterative solvers such as the Newton-Raphson method at each time step. Initial condition: $u_i^0 = u_0(x_i)$, boundary conditions: $u_0^n = u_N^n = 0$ for all n.

Example 8.5. Consider the singular fractional population model governed by the (q,τ)-Hadamard-type fractional time derivative:

$$ {}^{H}D^{\alpha,t}_{q,\tau} u(x,t) = {}^{H}D^{\beta,x}_{q,\tau} (D(x)u(x,t)) + r(x)u(x,t)\left(1 - \frac{u(x,t)}{K(x)}\right), $$

on the domain $x \in [1, 2],\ t \in [0, 1]$, with parameters:

$$ q = 0.8, \quad \tau = 1, \quad \alpha = 0.5, \quad \beta = 0.9. $$

Assume:

$$ D(x) = x, \quad r(x) = \sin(\pi x), \quad K(x) = 1 + x^2, \quad u_0(x) = \exp(-10(x-1.5)^2). $$

Let the spatial interval be divided into $N = 20$ nodes and time interval into $M = 50$ steps. Spatial grid: $h = 0.05$, time step: $\Delta t = 0.02$. We use the convolution weights previously mentioned to approximate the fractional operators. The fully discrete approach is used to compute the approximate solutions $u(x,t)$ in an iterative manner. The population spread is slowed down by the memory effect that the fractional time derivative introduces (see Fig.8). Growth is spatially heterogeneous due to the nonuniform environment (via $D(x)$, $r(x)$, $K(x)$). By altering the weighting in history dependency, the (q,τ)-structure provides tunable nonlocal behavior.

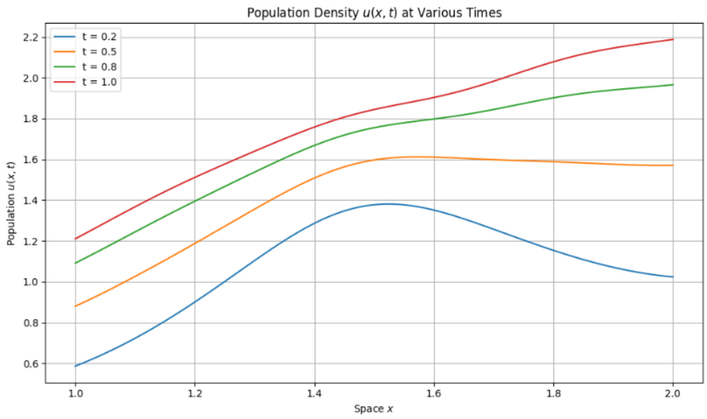

Figure 8: The plot shows the evolution of population density $u(x,t)$ at various time levels. The profile reflects memory effects and environmental heterogeneity governed by the $(q,1)$-Hadamard-type dynamics.

Example 8.6. We consider the following space-time fractional differential equation, when $\tau = 2$:

$$^{H}D^{\alpha,t}_{q,2}u(x,t) = {}^{H}D^{\beta,x}_{q,2}\left(D(x)u(x,t)\right) + r(x)u(x,t)\left(1 - \frac{u(x,t)}{K(x)}\right),$$

where $0 < \alpha, \beta < 1$, and $\tau = 2$. $x_j = a + jh$, $j = 0,1,\ldots,J$, with step size $h = \frac{b-a}{J}$, $t^n = n\Delta t$, $n = 0,1,\ldots,N$, with $\Delta t = \frac{T}{N}$, $u_j^n \approx u(x_j, t^n)$. The (q,τ)-Hadamard-type fractional derivative in time is approximated as:

$$^{H}D^{\alpha,t}_{q,2}u_j^n \approx \frac{1}{(\log q^{-1})^{\alpha}} \sum_{k=0}^{n} w_k^{(\alpha)} \frac{u_j^{n-k} - u_j^{n-k-1}}{\Delta t},$$

where the weights $w_k^{(\alpha)}$ can be defined by using the (q,τ)-Gamma function:

$$w_k^{(\alpha)} = \frac{\left(\log\left(\frac{t^{n-k+1}}{t^{n-k}}\right)\right)^{-\alpha}}{\Gamma_{q,2}(1-\alpha)}.$$

The (q,τ)-Hadamard-type space derivative is approximated as:

$$^{H}D^{\beta,x}_{q,2}\left(D(x_j)u_j^n\right) \approx \frac{1}{(\log q^{-1})^{\beta} h^{\beta}} \sum_{m=0}^{j} w_m^{(\beta)} \left(D_{j-m}u_{j-m}^n - D_{j-m-1}u_{j-m-1}^n\right).$$

The nonlinear reaction term is evaluated explicitly:

$$f_j^n = r(x_j)u_j^n \left(1 - \frac{u_j^n}{K(x_j)}\right).$$

Combining all parts, we obtain:

$$\frac{1}{(\log q^{-1})^\alpha} \sum_{k=0}^{n} w_k^{(\alpha)} \frac{u_j^{n-k} - u_j^{n-k-1}}{\Delta t} = \frac{1}{(\log q^{-1})^\beta h^\beta} \sum_{m=0}^{j} w_m^{(\beta)} \left(D_{j-m}u_{j-m}^n - D_{j-m-1}u_{j-m-1}^n\right) + f_j^n.$$

This provides an explicit time-stepping scheme to compute u_j^n at each time level n (see Fig.9).

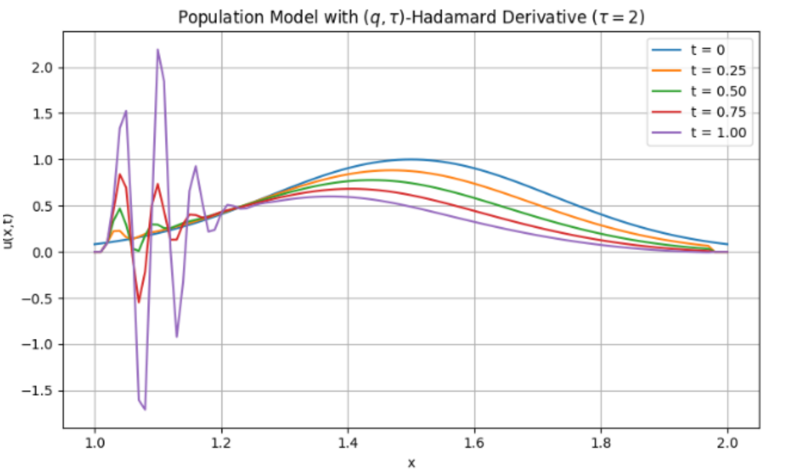

Figure 9: The plot shows the evolution of population density $u(x,t)$ at various time levels. The profile reflects memory effects and environmental heterogeneity governed by the $(q,2)$-Hadamard-type dynamics.

9. Conclusion

In this study, we used the (q,τ)-Hadamard-type fractional operators to design and analyze a fractional population dynamics model. By incorporating discrete scaling and nonlocal memory through the parameters q and τ, these operators extend the concept of conventional Hadamard derivatives. Both spatial heterogeneity and memory effects are important factors in nonhomogeneous environments, where the model accurately depicts the behavior of populations. We suggested a numerical method based on discretized Hadamard kernels and convolution quadrature. The framework's capacity to depict realistic ecological dynamics with memory was shown by numerical findings.

(q,τ)-Hadamard-type operators offer a versatile foundation for extending classical models to fractional and nonlocal regimes, as the results confirm. In Table 4, the Parameter τ in the (q,τ)-Hadamard Framework has the following benefits:

Table 4: Summary of the benefits of incorporating the parameter τ in fractional Hadamard-type operators.

Role of τ	**Benefit**
Kernel scaling	Adjusts singularity and decay of the kernel
Memory modeling	Controls fractional memory depth and strength
Numerical tuning	Enhances stability and convergence of numerical schemes
Generalization	Extends classical models to more flexible fractional structures
Physical modeling	Adapts to multiscale, anisotropic, or heterogeneous media
Functional theory	Enables richer analytic function subclasses and inclusions
Quantum extensions	Facilitates deformations in quantum groups and operators

References

[1] Jackson, F. H. (1909). XI.-On q-functions and a certain difference operator. Earth and Environmental Science Transactions of the Royal Society of Edinburgh, 46(2), 253-281.

[2] Kac, V. G., & Cheung, P. (2002). Quantum calculus (Vol. 113). New York: Springer.

[3] Ernst, T. (2012). A comprehensive treatment of q-calculus. Birkhauser.

[4] Attiya, A. A., Ibrahim, R. W., Hakami, A. H., Cho, N. E., & Yassen, M. F. (2025). Quantum-Fractal-Fractional Operator in a Complex Domain. Axioms, 14(1), 57.

[5] Tariq, M., Ntouyas, S. K., & Shaikh, A. A. (2023). A comprehensive review of the Hermite-Hadamard inequality pertaining to quantum calculus. Foundations, 3(2), 340-379.

[6] Fahad, H. M., Fernandez, A., Rehman, M. U., & Siddiqi, M. (2021). Tempered and Hadamard-type fractional calculus with respect to functions. Mediterranean Journal of Mathematics, 18(4), 143.

[7] Song, T. T., Wu, G. C., & Wei, J. L. (2022). Hadamard fractional calculus on time scales. Fractals, 30(07), 2250145.

[8] Ali, S. M., & Kassim, M. D. (2025). Sufficient Conditions for the Non-Existence of Global Solutions to Fractional Systems with Lower-Order Hadamard-Type Fractional Derivatives. Mathematics, 13(7), 1031.

[9] Luca, R., & Tudorache, A. (2025). Existence of solutions for a Hadamard fractional boundary value problem at resonance. Fractal and Fractional, 9(2), 119.

[10] Agrawal, P. N., & Baxhaku, B. (2025). Exponential sampling type neural network Kantorovich operators based on Hadamard fractional integral. Fractional Calculus and Applied Analysis, 1-36.

[11] Rivas, A., & Huelga, S. F. (2012). Open quantum systems (Vol. 10, pp. 978-3). Berlin: Springer.

[12] Banerjee, S., Banerjee, S., & Ahmad. (2018). Open quantum systems. Springer Singapore.

[13] Gao, H., Wang, K., Xiao, L., Nakagawa, M., Matsumoto, N., Qu, D., & Xue, P. (2024). Experimental observation of the Yang-Lee quantum criticality in open quantum systems. Physical Review Letters, 132(17), 176601.

[14] Kim, C. W., & Franco, I. (2024). General framework for quantifying dissipation pathways in open quantum systems. II. Numerical validation and the role of non-Markovianity. The Journal of Chemical Physics, 160(21).

[15] Vijayaram, S., & Balasubramaniam, P. (2024). Optimal control for nonlinear time-fractional Schrödinger equation: an application to quantum optics. Physica Scripta, 99(9), 095115.

[16] Kolokoltsov, V. N., & Shishkina, E. L. (2024). Fractional calculus for non-discrete signed measures. Mathematics, 12(18), 2804.

[17] Momani, S., & Ibrahim, R. W. (2025). Soliton propagation in optical metamaterials with nonlocal responses: A fractional calculus approach using (q, τ)-MittagLeffler functions. Partial Differential Equations in Applied Mathematics, 101305.

[18] Al-Shamayleh, A. S., & Ibrahim, R. W. (2025). Grapevine disease detection using (q, τ)-nabla calculus quantum deformation with deep learning features. MethodsX, 103619.

[19] Aldawish, I., & Ibrahim, R. W. (2025). Distributed-order (q, τ)-deformed Lévy processes and their spectral properties. Frontiers in Physics, 13, 1647182.

[20] Momani, S., & Ibrahim, R. W. (2025). Stability and entropy production in fractional bio-heat transport models via generalized (q, τ)-entropy. Frontiers in Applied Mathematics and Statistics, 11, 1643121.

[21] Momani, S., & Ibrahim, R. W. (2025). On the Mathematical Analysis of Generalized Quantum-Nabla Fractional Fluid Models with Dissipative Nonlinearities. Contemporary Mathematics, 7181-7213.

[22] Aldawish, I., & Ibrahim, R. W. (2025). Fractional topological invariants via (q, τ)-Gamma deformation in the non-Hermitian SSH model. AIP Advances, 15(11).

[23] Momani, S., & Ibrahim, R. W. (2025). Application of (q, τ)-Bernoulli Interpolation to the Spectral Solution of Quantum Differential Equations. International Journal of Differential Equations, (1), 4414882.

[24] De Moura, C. A., & Kubrusly, C. S. (2013). The courant-friedrichs lewy (cfl) condition. AMC, 10(12), 45-90.