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Lie symmetry analysis of time fractional Burgers equation with time-dependent coefficients

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Abstract

In this paper, Lie symmetry analysis is applied to study time fractional Burgers equation with time-dependent coefficients. Some Lie symmetries for the equation with some kinds of coefficients $f(t)$ and $g(t)$ are obtained. They are used to reduce the aimed equation with Riemann-Liouville fractional derivative to the fractional ordinary equation with Erdélyi-Kober fractional derivative. Then the power series method is applied to derive explicit power series solution for the reduced equation. For the power series solution, we not only provide a proof of its convergence but also conduct numerical simulations and analysis. In addition, the new conservation theorem and the generalization of Noether operators are developed to construct the conservation laws for the equation studied.

Keywords: Lie symmetry analysis, time fractional Burgers equation, Riemann-Liouville fractional derivative, Erdélyi-Kober fractional derivative, conservation laws.
2010 MSC: 76M60, 35G50, 37C79, 34K37.

1. Introduction

As a generalization of the classical calculus, fractional calculus can be traced back to the letter written by L'Hôspital to Leibniz in 1695. Since then, it has gradually gained the attention of mathematicians. Especially in recent decades, it has developed rapidly and been successfully applied in many fields of science and technology [1, 2, 3, 4]. Therefore, it is very important to find the solution of fractional differential equation. So far, there have been some numerical and analytical methods, such as Adomian decomposition method [5], finite difference method [6], homotopy perturbation method [7], the sub-equation method [8], the variational iteration method [9], Lie symmetry analysis method [10], invariant subspace method [11] and so on. Among them, Lie symmetry analysis method has received an increasing attention.

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Lie symmetry analysis method was founded by Norwegian mathematician Sophus Lie at the end of the nineteenth century and then further developed by some other mathematicians, such as Ovsiannikov [12], Olver [13], Ibragimov [14, 15, 16] and so on. As a modern method among many analytic techniques, Lie symmetry analysis has been extended to fractional differential equations (FDEs) by Gazizov et al. [10] in 2007. It was then effectively applied to various models of the FDEs occurring in different areas of applied science (see [17, 18, 19, 20, 21, 22, 23, 24, 25, 26]).

In this paper, Lie symmetry analysis method is applied to the following time fractional Burgers equation with time-dependent coefficients:

$$D_t^{\alpha}u + f(t)uu_x + g(t)u_{xx} = 0, \quad 0 < \alpha < 1, \tag{1.1}$$

which is widely used in physical and engineering fields, such as fluid mechanics, nonlinear acoustics, gas dynamics, etc. Recently, fractional Burgers-type equation was introduced and studied since the adding effect of the wall friction through the boundary layer can be modeled by fractional derivatives [27]. Various numerical and analytical methods have been applied to solve fractional Burgers-type equation [27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]. Among them, Lie symmetry analysis is an effective analytical method. In [35], Inc et al. used it to study the time fractional generalized Burgers-Huxley equation, which describes the interaction between reaction mechanism, convection effects and diffusion transport. In [36], Saqib et al. used it to study the time-fractional inviscid Burgers equation, which is the case of nonviscous fluids. In [37], Zhang used it to study the time-fractional coupled Burgers equation, which is a model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity. In [38], Yu used it to study the time-fractional Burgers equation with a delay.

The aim of this paper is to find all Lie symmetries admitted by Eq. (1.1) and construct the corresponding conserved vector for each symmetry by the new conservation theorem and the generalization of Noether operator. The obtained group generators are used to reduce Eq. (1.1) to time fractional ordinary differential equations, and some exact solutions are further derived. In addition, the convergence analysis and the numerical simulations for the power series solution are presented.

As we all know, there are many types of definitions for fractional derivative, such as Riemann-Liouville type, Caputo type, Weyl type and so on. This paper adopts Riemann-Liouville fractional derivative defined by

$${}_aD_t^{\alpha}f(t,x) = D_t^n\, {}_aI_t^{n-\alpha}f(t,x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(s,x)}{(t-s)^{\alpha-n+1}}ds, & n-1 < \alpha < n, n \in \mathbb{N}, \\ D_t^n f(t,x), & \alpha = n \in \mathbb{N}, \end{cases}$$

for $t > a$. We denote the operator ${}_0D_t^{\alpha}$ as D_t^{α} throughout this paper.

This paper is organized as follows. In Section 2, Lie symmetry analysis of Eq. (1.1) is presented. In Section 3, similarity reductions and exact solutions are obtained. In Section 4, the conserved vectors for all the symmetries admitted by Eq. (1.1) are constructed. The conclusion is given in the last section.

2. Lie symmetry analysis of Eq. (1.1)

Consider time fractional Burgers equation with time-dependent coefficients (1.1), which is assumed to be invariant under the one-parameter (ϵ) Lie group of continuous point transformations, i.e.

$$\begin{aligned} t^* &= t+\epsilon\tau(t,x,u)+o(\epsilon),\\ x^* &= x+\epsilon\xi(t,x,u)+o(\epsilon),\\ u^* &= u+\epsilon\eta(t,x,u)+o(\epsilon),\\ D_{x^*}u^* &= D_x u+\epsilon\eta^x+o(\epsilon),\\ D^2_{x^*}u^* &= D^2_x u+\epsilon\eta^{xx}+o(\epsilon),\\ D^\alpha_{t^*}u^* &= D^\alpha_t u+\epsilon\eta^{\alpha,t}+o(\epsilon), \end{aligned} \tag{2.1}$$

where τ, ξ and η are infinitesimals and η^x, η^{xx} and $\eta^{\alpha,t}$ are the corresponding prolongations of order 1, 2 and α respectively.

The corresponding group generator is defined by

$$X=\tau(t,x,u)\frac{\partial}{\partial t}+\xi(t,x,u)\frac{\partial}{\partial x}+\eta(t,x,u)\frac{\partial}{\partial u}. \tag{2.2}$$

So the prolongation of the above group generator X has the form

$$Pr^{(\alpha,2)}X=X+\eta^{\alpha,t}\frac{\partial}{\partial u_t^\alpha}+\eta^x\frac{\partial}{\partial u_x}+\eta^{xx}\frac{\partial}{\partial u_{xx}}, \tag{2.3}$$

where

$$\begin{aligned} \eta^x =&D_x\eta-u_tD_x\tau-u_xD_x\xi=\eta_x+(\eta_u-\xi_x)u_x-\tau_xu_t-\xi_u(u_x)^2-\tau_uu_xu_t,\\ \eta^{xx} =&D_x\eta^x-u_{xt}D_x\tau-u_{xx}D_x\xi=\eta_{xx}+(2\eta_{xu}-\xi_{xx})u_x-\tau_{xx}u_t+(\eta_{uu}-2\xi_{xu})(u_x)^2\\ &-2\tau_{xu}u_xu_t-\xi_{uu}(u_x)^3-\tau_{uu}(u_x)^2u_t+(\eta_u-2\xi_x)u_{xx}-2\tau_xu_{xt}-3\xi_uu_{xx}u_x\\ &-\tau_uu_{xx}u_t-2\tau_uu_{xt}u_x, \end{aligned}$$

and

$$\begin{aligned} \eta^{\alpha,t} =&D^\alpha_t\eta+\xi D^\alpha_t u_x-D^\alpha_t(\xi u_x)+D^\alpha_t(uD_t\tau)-D^{\alpha+1}_t(\tau u)+\tau D^{\alpha+1}_t u\\ =&\frac{\partial^\alpha\eta}{\partial t^\alpha}+(\eta_u-\alpha D_t\tau)\frac{\partial^\alpha u}{\partial t^\alpha}-u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}-\sum_{n=1}^{\infty}\binom{\alpha}{n}D^n_t\xi D^{\alpha-n}_t u_x\\ &+\sum_{n=1}^{\infty}[\binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n}-\binom{\alpha}{n+1}D^{n+1}_t\tau]D^{\alpha-n}_t u+\mu, \end{aligned}$$

with

$$\mu=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-u)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m u^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.$$

Note that D_t and D_x are the total derivative with respect to t and x respectively.

Remark 2.1. Lie symmetry transformations (2.1) should conserve the structure of the Riemann-Liouville fractional derivative operator, of which the lower limit in the integral is fixed. Therefore, the manifold $t=0$ should be invariant with respect to such transformations. The invariance condition arrives at

$$\tau(t,x,u)|_{t=0}=0. \tag{2.4}$$

Remark 2.2. The derivatives $\partial^k\eta/\partial u^k, k\geqslant 2$ exist in the expression of μ. Therefore, if the infinitesimal η be linear with respect to the variable u then $\mu=0$, that is,

$$\frac{\partial^2\eta}{\partial u^2}=0. \tag{2.5}$$

Lie symmetry transformations (2.1) are admitted by Eq. (1.1), if the following invariant criterion holds:

$$\mathrm{Pr}^{(\alpha,2)}X\big(D_t^\alpha u+f(t)uu_x+g(t)u_{xx}\big)|_{(1.1)}=0, \tag{2.6}$$

which can be rewritten as

$$\big(\eta^{\alpha,t}+f(t)u\eta^x+g(t)\eta^{xx}+f'(t)\tau uu_x+f(t)\eta u_x+g'(t)\tau u_{xx}\big)|_{(1.1)}=0. \tag{2.7}$$

Putting $\eta^{\alpha,t}$, η^x and η^{xx} into (2.7), and letting coefficients of various derivatives of u to be zero, we can obtain the over-determined system of differential equations as follows:

$$\tau_x=\tau_u=\xi_t=\xi_u=\eta_{uu}=0, \tag{2.8}$$

$$\binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n}-\binom{\alpha}{n+1}D_t^{n+1}(\tau)=0,\ \ n\in N, \tag{2.9}$$

$$(\alpha\tau_t-2\xi_x)g(t)+\tau g'(t)=0, \tag{2.10}$$

$$\tau uf'(t)+f(t)(\eta+\alpha u\tau_t-u\xi_x)+g(t)(2\eta_{ux}-\xi_{xx})=0, \tag{2.11}$$

$$\frac{\partial^\alpha\eta}{\partial t^\alpha}-u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}+f(t)u\eta_x+g(t)\eta_{xx}=0. \tag{2.12}$$

From the equations (2.8)–(2.10) and the conditions (2.4)–(2.5), we can obtain

$$\tau=c_1t,\ \ \xi=\frac{n+\alpha}{2}c_1x+c_2,\ \ \eta=\phi(x)u+\varphi(t,x),\ \ g(t)=kt^n. \tag{2.13}$$

Then (2.11) and (2.12) become

$$c_1tuf'(t)+f(t)(\phi(x)u+\varphi(t,x)-\frac{n+\alpha}{2}c_1u)+2kt^n\phi'(x)=0, \tag{2.14}$$

$$(\frac{\partial^\alpha\varphi}{\partial t^\alpha}+f(t)u\varphi_x+g(t)\varphi_{xx})+u^2f(t)\phi'(x)+2kt^nu\phi''(x)=0. \tag{2.15}$$

From (2.15), $\varphi(t,x)$ is an arbitrary solution of Eq. (1.1), and

$$uf(t)\phi'(x)+2kt^n\phi''(x)=0. \tag{2.16}$$

So there are the following two cases from (2.14) and (2.16).

Case 1: $f(t) = 0, \ k = 0$

In this case, we can obtain the following infinitesimals:

$$\tau = c_1 t, \quad \xi = \frac{n+\alpha}{2} c_1 x + c_2, \quad \eta = \phi(x)u + \varphi(t,x). \tag{2.17}$$

Then the group generators admitted by Eq.(1.1) are constructed as

$$X_1 = t\frac{\partial}{\partial t} + \frac{n+\alpha}{2} x \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \phi(x)u\frac{\partial}{\partial u}, \quad X_\infty = \varphi(t,x)\frac{\partial}{\partial u}. \tag{2.18}$$

However, Eq.(1.1) becomes

$$D_t^\alpha u = 0, \tag{2.19}$$

which has no practical significance.

Case 2: $\phi'(x) = 0$

In this case, $\eta = c_3 u + \varphi(t,x)$, and (2.14) becomes

$$c_1 t u f'(t) + f(t)(c_3 u - \frac{n+\alpha}{2} c_1 u + \varphi(t,x)) = 0. \tag{2.20}$$

If $\varphi(t,x) = 0$, then (2.20) becomes

$$c_1 t f'(t) + (c_3 - \frac{n+\alpha}{2} c_1) f(t) = 0, \tag{2.21}$$

of which the solutions are

$$f(t) = l t^m, \quad \text{with} \ \ c_3 = \frac{n-2m+\alpha}{2} c_1. \tag{2.22}$$

Now, the obtained infinitesimals are

$$\tau = c_1 t, \quad \xi = \frac{n+\alpha}{2} c_1 x + c_2, \quad \eta = \frac{n-2m+\alpha}{2} c_1 u, \tag{2.23}$$

and the corresponding group generators are

$$X_1 = t\frac{\partial}{\partial t} + \frac{n+\alpha}{2} x \frac{\partial}{\partial x} + \frac{n-2m+\alpha}{2} u \frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial x}. \tag{2.24}$$

Then Eq.(1.1) becomes

$$D_t^\alpha u + l t^m u u_x + k t^n u_{xx} = 0, \tag{2.25}$$

where k, l, m and n are arbitrary constants.

3. Reductions and exact solutions of Eq. (1.1)

3.1. $X_1 = t\frac{\partial}{\partial t} + \frac{n+\alpha}{2}x\frac{\partial}{\partial x} + \frac{n-2m+\alpha}{2}u\frac{\partial}{\partial u}$

The characteristic equation of X_1 is

$$\frac{dt}{t} = \frac{2dx}{n+\alpha} = \frac{2du}{n-2m+\alpha}, \tag{3.1}$$

from which, we obtain the similarity variables $xt^{-\frac{n+\alpha}{2}}$ and $ut^{-\frac{n-2m+\alpha}{2}}$. So we get group invariant solution

$$u(t,x) = t^{\frac{n-2m+\alpha}{2}}h(\omega), \quad \omega = xt^{-\frac{n+\alpha}{2}}. \tag{3.2}$$

Theorem 3.1. *The similarity transformation* $u(t,x) = t^{\frac{n-2m+\alpha}{2}}h(\omega)$ *with the similarity variable* $\omega = xt^{-\frac{n+\alpha}{2}}$ *reduce Eq. (2.25) to the fractional ordinary differential equation given by*

$$(\mathcal{P}_{\frac{2}{n+\alpha}}^{1+\frac{n-2m-\alpha}{2},\alpha}h)(\omega) + lh(\omega)h'(\omega) + kh''(\omega) = 0, \tag{3.3}$$

where $(\mathcal{P}_\delta^{\iota,\kappa})$ *is the left-hand Erdélyi-Kober fractional differential operator defined by*

$$(\mathcal{P}_\delta^{\iota,\kappa}\psi)(\omega) := \prod_j^{m-1}(\iota + j - \frac{1}{\delta}\omega\frac{d}{d\omega})(\mathcal{K}_\delta^{\iota+\kappa,m-\kappa}\psi)(\omega), \quad \omega > 0,\ \delta > 0,\ \kappa > 0, \tag{3.4}$$

$$m = \begin{cases} [\kappa] + 1, & \text{if } \kappa \notin \mathbb{N}, \\ \kappa, & \text{if } \kappa \in \mathbb{N}, \end{cases}$$

where

$$(\mathcal{K}_\delta^{\iota,\kappa}\psi)(\omega) := \begin{cases} \frac{1}{\Gamma(\kappa)}\int_1^\infty (s-1)^{\kappa-1}s^{-(\iota+\kappa)}\psi(\omega s^{\frac{1}{\delta}})ds, & \text{if } \kappa > 0, \\ \psi(\omega), & \text{if } \kappa = 0, \end{cases} \tag{3.5}$$

is the left-hand Erdélyi-Kober fractional integral operator.

Proof. For $0 < \alpha < 1$, the Riemann-Liouville time fractional derivative of $u(t,x)$ can be obtained as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\alpha}{\partial t^\alpha}(h(\omega)) = \frac{\partial}{\partial t}\Big[\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}s^{\frac{n-2m+\alpha}{2}}h(xs^{-\frac{n+\alpha}{2}})ds\Big].$$

Assuming $r = \frac{t}{s}$, we have

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} =& \frac{\partial}{\partial t}\Big[\frac{t^{1+\frac{n-2m-\alpha}{2}}}{\Gamma(1-\alpha)}\int_1^\infty (r-1)^{-\alpha}r^{-\frac{n-2m-\alpha}{2}-2}h(\omega r^{\frac{n+\alpha}{2}})dr\Big] \\ =& \frac{\partial}{\partial t}\Big[t^{1+\frac{n-2m-\alpha}{2}}(\mathcal{K}_{\frac{2}{n+\alpha}}^{1+\frac{n-2m+\alpha}{2},1-\alpha}h)(\omega)\Big].\end{aligned}$$

Because of $\omega = xt^{-\frac{n+\alpha}{2}}$, the following relation holds:

$$t\frac{\partial}{\partial t}\psi(\omega) = tx(-\frac{n+\alpha}{2})t^{-\frac{n+\alpha}{2}-1}\psi'(\omega) = -\frac{n+\alpha}{2}\omega\frac{d}{d\omega}\psi(\omega).$$

Hence, we arrive at

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} =& t^{\frac{n-2m-\alpha}{2}}\Big[(1+\frac{n-2m-\alpha}{2}-\frac{n+\alpha}{2}\omega\frac{d}{d\omega})(\mathcal{K}^{1+\frac{n-2m+\alpha}{2},1-\alpha}_{\frac{2}{n+\alpha}}h)(\omega)\Big]\\ =& t^{\frac{n-2m-\alpha}{2}}(\mathcal{P}^{1+\frac{n-2m-\alpha}{2},\alpha}_{\frac{2}{n+\alpha}}h)(\omega).\end{aligned}$$

With $u=t^{\frac{n-2m+\alpha}{2}}h(\omega)$, $\omega=xt^{-\frac{n+\alpha}{2}}$, we obtain

$$lt^m uu_x+kt^n u_{xx}=t^{\frac{n-2m-\alpha}{2}}\big(lh(\omega)h'(\omega)+kh''(\omega)\big).$$

This completes the proof. □

Next we use the power series method to derive the power series solutions of (3.3). Assuming

$$h(\omega)=\sum_{i=0}^{\infty}a_i\omega^i,\quad \omega=xt^{-\frac{n+\alpha}{2}}, \tag{3.6}$$

where a_i are constants to be known later, from the definition of Erdélyi-Kober fractional integral operator, we get

$$\begin{aligned}(\mathcal{K}^{1+\frac{n-2m+\alpha}{2},1-\alpha}_{\frac{2}{n+\alpha}}h)(\omega) =& \frac{1}{\Gamma(1-\alpha)}\int_1^\infty (s-1)^{-\alpha}s^{-\frac{n-2m-\alpha}{2}-2}h(\omega s^{\frac{n+\alpha}{2}})ds\\ =& \frac{1}{\Gamma(1-\alpha)}\int_1^\infty (s-1)^{-\alpha}s^{-\frac{n-2m-\alpha}{2}-2}\sum_{i=0}^{\infty}(a_i\omega^i s^{\frac{i(n+\alpha)}{2}})ds\\ =& \sum_{i=0}^{\infty}a_i\omega^i\Big[\frac{1}{\Gamma(1-\alpha)}\int_1^\infty (s-1)^{-\alpha}s^{\frac{i(n+\alpha)-(n-2m-\alpha)}{2}-2}ds\Big].\end{aligned}$$

Because of Beta function $B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}ds$, assuming $t=\frac{1}{x}$, we have

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx=\int_1^\infty (t-1)^{q-1}t^{-(p+q)}dt.$$

So

$$\begin{aligned}(\mathcal{K}^{1+\frac{n-2m+\alpha}{2},1-\alpha}_{\frac{2}{n+\alpha}}h)(\omega) =& \sum_{i=0}^{\infty}a_i\omega^i\frac{B(1-\frac{i(n+\alpha)-(n-2m+\alpha)}{2},1-\alpha)}{\Gamma(1-\alpha)}\\ =& \sum_{i=0}^{\infty}\frac{\Gamma(1-\frac{i(n+\alpha)-(n-2m+\alpha)}{2})}{\Gamma(2-\frac{i(n+\alpha)-(n-2m-\alpha)}{2})}a_i\omega^i,\end{aligned}$$

from which, we get

$$\begin{aligned}&(\mathcal{P}^{1+\frac{n-2m-\alpha}{2},\alpha}_{\frac{2}{n+\alpha}}h)(\omega)=(1+\frac{n-2m-\alpha}{2}-\frac{n+\alpha}{2}\omega\frac{d}{d\omega})(\mathcal{K}^{1+\frac{n-2m+\alpha}{2},1-\alpha}_{\frac{2}{n+\alpha}}h)(\omega)\\ &=(1+\frac{n-2m-\alpha}{2}-\frac{n+\alpha}{2}\omega\frac{d}{d\omega})(\sum_{i=0}^{\infty}\frac{\Gamma(1-\frac{i(n+\alpha)-(n-2m+\alpha)}{2})}{\Gamma(2-\frac{i(n+\alpha)-(n-2m-\alpha)}{2})}a_i\omega^i)\\ &=\sum_{i=0}^{\infty}\frac{\Gamma(1+\frac{n-2m+\alpha}{2}-\frac{i(n+\alpha)}{2})}{\Gamma(1+\frac{n-2m-\alpha}{2}-\frac{i(n+\alpha)}{2})}a_i\omega^i.\end{aligned} \tag{3.7}$$

From (3.6), we have

$$h'(\omega)=\sum_{i=0}^{\infty}(i+1)a_{i+1}\omega^{i},\ \ h''(\omega)=\sum_{i=0}^{\infty}(i+2)(i+1)a_{i+2}\omega^{i}. \tag{3.8}$$

Substituting (3.6)-(3.8) into (3.3) arrives at the following equation:

$$\sum_{i=0}^{\infty}\frac{\Gamma(1+\frac{n-2m+\alpha}{2}-\frac{i(n+\alpha)}{2})}{\Gamma(1+\frac{n-2m-\alpha}{2}-\frac{i(n+\alpha)}{2})}a_i\omega^i=\sum_{i=0}^{\infty}\Big(l\sum_{p+q=i}(q+1)a_pa_{q+1}+k(i+2)(i+1)a_{i+2}\Big)\omega^i. \tag{3.9}$$

We can equate the coefficients of different powers of ω to obtain the explicit expressions of a_i as $a_0=h(0)$, $a_1=h'(0)$ and

$$a_{i+2}=\frac{1}{k(i+2)(i+1)}\Big[\frac{\Gamma(1+\frac{n-2m+\alpha}{2}-\frac{i(n+\alpha)}{2})}{\Gamma(1+\frac{n-2m-\alpha}{2}-\frac{i(n+\alpha)}{2})}a_i-l\sum_{p+q=i}(q+1)a_pa_{q+1}\Big],i\geqslant 0. \tag{3.10}$$

Thus we get a power series solution of Eq. (2.25) in the form

$$\begin{aligned}u(t,x)=&t^{\frac{n-2m+\alpha}{2}}\Big\{a_0+a_1xt^{-\frac{n+\alpha}{2}}+\frac{1}{2k}\Big[\frac{\Gamma(1+\frac{n-2m+\alpha}{2})}{\Gamma(1+\frac{n-2m-\alpha}{2})}a_0-la_0a_1\Big]x^2t^{-(n+\alpha)}\\&+\sum_{i=1}^{\infty}\frac{1}{k(i+2)(i+1)}\Big[\frac{\Gamma(1+\frac{n-2m+\alpha}{2}-\frac{i(n+\alpha)}{2})}{\Gamma(1+\frac{n-2m-\alpha}{2}-\frac{i(n+\alpha)}{2})}a_i\\&-l\sum_{p+q=i}(q+1)a_pa_{q+1}\Big]x^{i+2}t^{-\frac{(i+2)(n+\alpha)}{2}}\Big\}.\end{aligned} \tag{3.11}$$

Theorem 3.2. *The power series solution (3.11) is convergent in a neighborhood of the point* $(0,|a_0|)$.

Proof. Considering Eq. (3.10), we can obtain

$$|a_{i+2}|\leqslant|\frac{1}{k}|\Big[\frac{|\Gamma(1+\frac{n-2m+\alpha}{2}-\frac{i(n+\alpha)}{2})|}{|\Gamma(1+\frac{n-2m-\alpha}{2}-\frac{i(n+\alpha)}{2})|}|a_i|+|l|\sum_{p+q=i}|a_p||a_{q+1}|\Big],\ \ i\geqslant 0. \tag{3.12}$$

From the properties of Γ function, it can easily be found that $\frac{|\Gamma(1+\frac{n-2m+\alpha}{2}-\frac{i(n+\alpha)}{2})|}{|\Gamma(1+\frac{n-2m-\alpha}{2}-\frac{i(n+\alpha)}{2})|}\leqslant 1$ for arbitrary i. Thus, (3.12) can be written as

$$|a_{i+2}|\leqslant M\Big(|a_i|+\sum_{p+q=i}|a_p||a_{q+1}|\Big),\ \ i\geqslant 0, \tag{3.13}$$

where $M=max\{1,|\frac{1}{k}|,|\frac{l}{k}|\}$.

Consider another power series

$$B(\omega)=\sum_{i=0}^{\infty}b_i\omega^i,\ \ \omega=xt^{-\frac{n+\alpha}{2}}, \tag{3.14}$$

where $b_0 = |a_0|$, $b_1 = |a_1|$ and

$$b_{i+2} = M(b_i + \sum_{p+q=i} b_p b_{q+1}), \quad i \geqslant 0. \tag{3.15}$$

Therefore, it is easily seen that $|a_i| \leqslant b_i$ for $i = 0, 1, 2, \ldots$, that is, the power series (3.14) is the majorant series of (3.6). We next show that the power series (3.14) is convergent. By simple calculation, we can get

$$B(\omega) = b_0 + b_1\omega + M\big(B(\omega)\omega^2 + B(\omega)(B(\omega) - b_0)\omega\big). \tag{3.16}$$

Consider the following implicit function with respect to the independent variable ω:

$$\Psi(\omega, B) = B - b_0 - b_1\omega - M\big(B\omega^2 + B(B - b_0)\omega\big), \tag{3.17}$$

it can be seen that $\Psi(\omega, B)$ is analytic in a neighborhood of $(0, b_0)$, where $\Psi(0, b_0) = 0$ and $\frac{\partial}{\partial B}\Psi(0, b_0) \neq 0$. Then, by implicit function theorem [39], one can see that the power series (3.14) is analytic in neighborhood of the point $(0, b_0)$. This implies that the power series solution (3.11) is convergent in a neighborhood of the point $(0, |a_0|)$ and with a positive radius. This completes the proof. □

Graphical representations of the power series solution (3.11) are given in Figures 1–2, where the coefficients of the first five items under different conditions are listed in Tables 1–2. The behaviors of the solution depend on the fractional order α and the time-dependent coefficients $f(t) = lt^m$, $g(t) = kt^n$. For example, in Figure 1, we select different fractional orders for the given time-dependent coefficients, while in Figure 2, we choose different time-dependent coefficients for a given fractional order.

Table 1: Some of a_i for different fractional orders

	a_0	a_1	a_2	a_3	a_4	a_5
$\alpha = 0.4$	1	1	-0.1056754387	-0.07554876537	0.2520702348	-0.2775967504
$\alpha = 0.6$	1	1	-0.1543018338	-0.1760489574	0.1454474914	0.3436389408
$\alpha = 0.8$	1	1	-0.2020989674	-0.2768112675	-0.0142757351	-0.01731486175

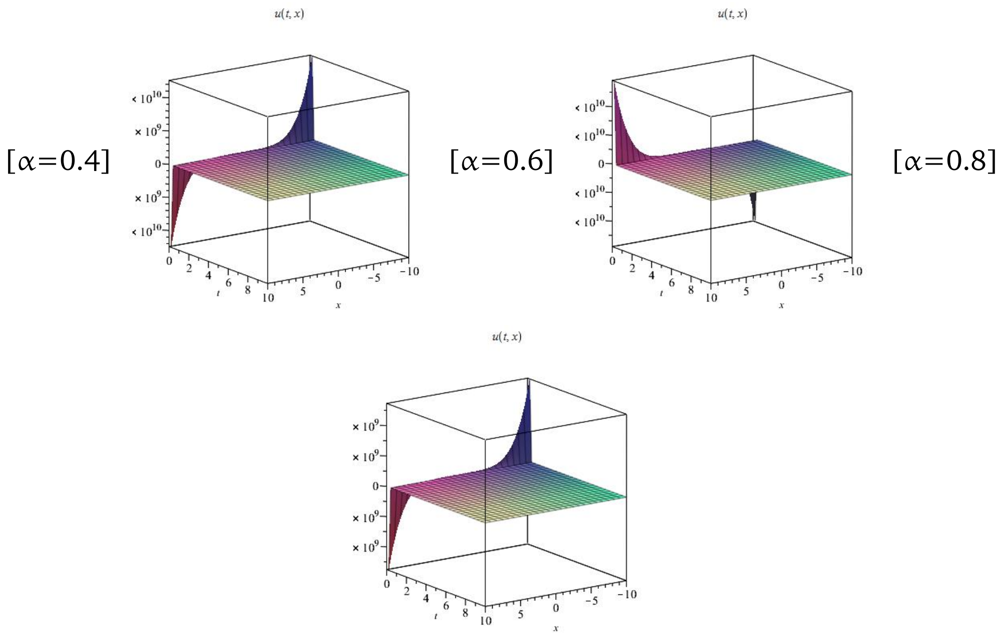

Figure 1: Numerical simulation of the power series solution (3.11) with $k = l = 1,\ m = 1.5,\ n = 3$.

Table 2: Some of a_i for different values of parameter m, n

	a_0	a_1	a_2	a_3	a_4	a_5
$m = 2.5, n = 4.5$	1	1	-0.3013566646	-0.2922158646	0.4271865552	-0.3606146385
$m = 3.5, n = 5.5$	1	1	-0.5822807636	-0.2663749241	-0.1271829654	-0.3416490687
$m = 4.5, n = 2.5$	1	1	-3.082363360	0.5040284032	0.9641761994	-1.568276225

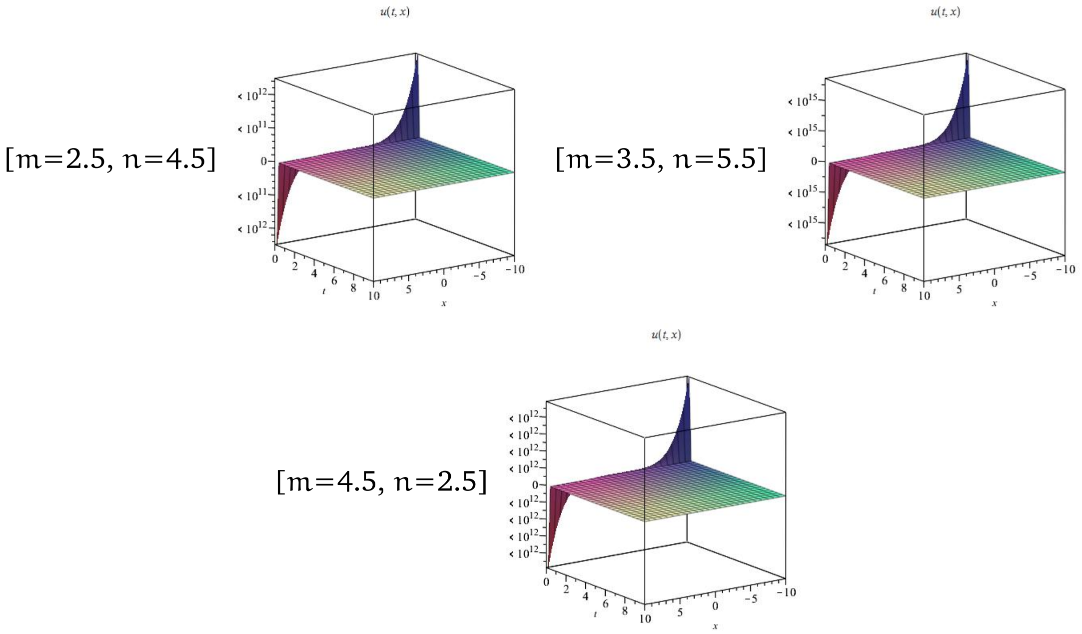

Figure 2: Numerical simulation of the power series solution (3.11) with $k = l = 1$, $\alpha = 0.75$.

3.2. $X_2 = \frac{\partial}{\partial x}$

The characteristic equation of X_2 is

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0}, \tag{3.18}$$

from which, we obtain the similarity variables t and u. So we get group invariant solution

$$u = h(\omega), \quad \omega = t. \tag{3.19}$$

Then, Eq. (1.1) becomes

$$D_\omega^\alpha h(\omega) = 0, \tag{3.20}$$

from which, the trivial solution of Eq. (1.1) can be obtained as

$$u = \frac{c}{\Gamma(\alpha)} t^{\alpha-1}, \tag{3.21}$$

where c is an arbitrary constant.

4. Conservation laws of Eq. (1.1)

In this section, we will construct conservation laws of Eq. (1.1) with $f(t) = lt^m$ and $g(t) = kt^n$ by using the generalization of the Noether operators and the new conservation theorem [40, 41].

The formal Lagrangian for Eq. (1.1),

$$F = D_t^\alpha u + lt^m u u_x + kt^n u_{xx} = 0, \tag{4.1}$$

is given by

$$\mathcal{L} = v(t,x)F = v(t,x)(D_t^\alpha u + lt^m uu_x + kt^n u_{xx}), \tag{4.2}$$

where $v(t,x)$ is a new dependent variable. The Euler-Lagrange operator is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial(D_t^\alpha u)} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}}, \tag{4.3}$$

where $(D_t^\alpha)^*$ is the adjoint operator of D_t^α. It is defined by

$$(D_t^\alpha)^* = (-1)^n \,{}_tJ_T^{n-\alpha}(D_t^n) \equiv {}_t^cD_T^\alpha, \tag{4.4}$$

where ${}_t^cD_T^\alpha$ is the right-sided of Caputo fractional derivative, i.e.,

$${}_t^cD_T^\alpha f(t,x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_t^T \frac{1}{(t-s)^{\alpha-n+1}}\frac{\partial^n}{\partial s^n}f(s,x)ds, & n-1<\alpha<n, n\in\mathbb{N}, \\ D_t^n f(t,x), & \alpha = n\in\mathbb{N}. \end{cases}$$

The adjoint equation of Eq. (1.1) is given by

$$F^* = \frac{\delta\mathcal{L}}{\delta u} = (D_t^\alpha)^* v - lt^m uv_x + kt^n v_{xx} = 0. \tag{4.5}$$

Next we will use the above adjoint equation and the new conservation theorem to construct conservation laws of Eq. (1.1). From the classical definition of the conservation laws, a vector $C=(C^t, C^x)$ is called a conserved vector for the governing equation if it satisfies the conservation equation $[D_tC^t + D_xC^x]_{F=0} = 0$. By using Noether theorem the components of conserved vector can be obtained.

Firstly, from the fundamental operator identity, i.e.,

$$Pr^{(\alpha,2)}X + D_t\tau\cdot\mathcal{I} + D_x\xi\cdot\mathcal{I} = W\cdot\frac{\delta}{\delta u} + D_t\mathcal{N}^t + D_x\mathcal{N}^x, \tag{4.6}$$

where $Pr^{(\alpha,2)}X$ is mentioned in (2.3), $\mathcal{I}$ is the identity operator and $W = \eta - \tau u_t - \xi u_x$ is the characteristic for group generator X, we can get the Noether operators as follows:

$$\mathcal{N}^t = \tau\mathcal{I} + \sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W)D_t^k\frac{\partial}{\partial(D_t^\alpha u)} - (-1)^n J(W, D_t^n\frac{\partial}{\partial(D_t^\alpha u)}), \tag{4.7}$$

$$\mathcal{N}^x = \xi\mathcal{I} + W(\frac{\partial}{\partial u_x} - D_x\frac{\partial}{\partial u_{xx}}) + D_x(W)\frac{\partial}{\partial u_{xx}}, \tag{4.8}$$

where $n=[\alpha]+1$, and J is given by

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T \frac{f(\tau,x)g(\theta,x)}{(\theta-\tau)^{\alpha+1-n}}d\theta d\tau. \tag{4.9}$$

The components of conserved vector are defined by $C^t = \mathcal{N}^t\mathcal{L}$, $C^x = \mathcal{N}^x\mathcal{L}$.

Case 1: $X_1 = t\frac{\partial}{\partial t} + \frac{n+\alpha}{2}x\frac{\partial}{\partial x} + \frac{n-2m+\alpha}{2}u\frac{\partial}{\partial u}$

The characteristic of X_1 is

$$W=\frac{n-2m+\alpha}{2}u-tu_t-\frac{n+\alpha}{2}xu_x. \tag{4.10}$$

Therefore, for $0<\alpha<1$,

$$\begin{aligned}C^t=&vD_t^{\alpha-1}(W)+J(W,v_t)=vD_t^{\alpha-1}(\frac{n-2m+\alpha}{2}u-tu_t-\frac{n+\alpha}{2}xu_x)\\&+J(\frac{n-2m+\alpha}{2}u-tu_t-\frac{n+\alpha}{2}xu_x,v_t),\end{aligned} \tag{4.11}$$

$$\begin{aligned}C^x=&(lt^muv-kt^nv_x)W+kt^nvD_x(W)=(lt^muv-kt^nv_x)(\frac{n-2m+\alpha}{2}u-tu_t\\&-\frac{n+\alpha}{2}xu_x)+kt^nv(\frac{n-2m+\alpha}{2}u_x-tu_{xt}-\frac{n+\alpha}{2}u_x-\frac{n+\alpha}{2}xu_{xx}).\end{aligned} \tag{4.12}$$

Case 2: $X_2=\frac{\partial}{\partial x}$

The characteristic of X_2 is

$$W=-u_x. \tag{4.13}$$

Therefore, for $0<\alpha<1$,

$$C^t=vD_t^{\alpha-1}(W)+J(W,v_t)=-vD_t^{\alpha-1}(u_x)-J(u_x,v_t), \tag{4.14}$$

$$C^x=(lt^muv-kt^nv_x)W+kt^nvD_x(W)=-(lt^muv-kt^nv_x)u_x-kt^nvu_{xx}. \tag{4.15}$$

5. Conclusion

For time fractional Burgers equation with time-dependent coefficients, by solving the obtained over-determined system, we get all the Lie symmetries for the equation with some special form of coefficient functions $f(t)$ and $g(t)$. Then the Lie symmetries are used to reduce the aimed equation to some fractional ordinary equations. Furthermore, the power series solution and its numerical simulation are given. The conservation laws for the equation are obtained by Ibragimov's method [40, 41]. Future research directions involve applying the existence and uniqueness results of solutions for fractional differential equations [42], as well as outcomes related to fractional differential equations under periodic and anti-periodic boundary conditions [43, 44], to time fractional Burgers equation. Furthermore, it is essential to extend the Lie symmetry analysis method to more significant forms of nonlinear fractional differential equations.

Declarations

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability statement

All data generated or analysed during this study are included in this published article.

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