Some weighted Simpson type inequalities for differentiable s -convex functions and their applications

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Abstract

In this study, by using a new identity we establish some new Simpson type inequalities for differentiable s -convex functions in the second sense. Various special cases have been studied in details. Also, in order to illustrate the efficient of our main results, some applications to special means and weighted Simpson quadrature formula are given. The obtained results generalize and refine certain known results. At the end, a brief conclusion is given as well.

Keywords: Simpson inequality, weighted function, s -convex functions, Hölder inequality, weighted Simpson quadrature formula.

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1. Introduction

Definition 1.1. ([1]) Let I be an interval of real numbers. A function $\psi : I \rightarrow \mathbb{R}$ is said to be convex, if for all $\lambda_1, \lambda_2 \in I$ and all $\chi \in [0, 1]$, we have

$$\psi(\chi\lambda_1 + (1 - \chi)\lambda_2) \leq \chi\psi(\lambda_1) + (1 - \chi)\psi(\lambda_2).$$

The concept of convex functions has been also generalized in diverse manners. One of them is the so-called s -convex function defined as follows:

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Definition 1.2. ([2]) A nonnegative function $\psi : I \subset [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex in the second sense for some fixed $s \in (0, 1]$, if

$$\psi(\chi\lambda_1 + (1 - \chi)\lambda_2) \leq \chi^s \psi(\lambda_1) + (1 - \chi)^s \psi(\lambda_2)$$

holds for all $\lambda_1, \lambda_2 \in I$ and $\chi \in [0, 1]$.

The following inequality is known in the literature as Simpson's inequality.

Theorem 1.3. Let ψ be four times continuously differentiable function on (λ_1, λ_2) and $\|\psi^{(4)}\|_\infty := \sup_{x \in (\lambda_1, \lambda_2)} |\psi^{(4)}(x)| < \infty$, then

$$\left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \psi(u) du \right| \leq \frac{1}{2880} \|\psi^{(4)}\|_\infty (\lambda_2 - \lambda_1)^4.$$

In recent years, many researchers have studied the error estimates of Simpson's inequality, in order to establish new refinements, generalizations as well as new Simpson-type inequalities for more details we refer readers [3, 4, 5, 6, 7, 8, 9]. For an overview, the reader should see also the following literatures on integral inequalities, [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30] and the references therein. The following notations will be used in sequel. We denote, respectively I° the interior of I and $\mathcal{L}[\lambda_1, \lambda_2]$ the set of all integrable functions on $[\lambda_1, \lambda_2]$. In [10], Alomari et al. established the following Simpson type inequalities for s -convex functions.

Theorem 1.4. Let $\psi : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$, where $\lambda_1, \lambda_2 \in I^\circ$ with $\lambda_1 < \lambda_2$. If $|\psi'|$ is s -convex on $[\lambda_1, \lambda_2]$ for some fixed $s \in (0, 1]$, then the following inequality holds:

$$\left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \psi(u) du \right| \leq (\lambda_2 - \lambda_1) \frac{6^{-s} - 9 \times 2^{-s} + 5^{s+2} \times 6^{-s} + 3s - 12}{18(s^2 + 3s + 2)} [|\psi'(\lambda_1)| + |\psi'(\lambda_2)|].$$

Theorem 1.5. Let $\psi : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$, where $\lambda_1, \lambda_2 \in I^\circ$ with $\lambda_1 < \lambda_2$. If $|\psi'|^{p/(p-1)}$ is s -convex on $[\lambda_1, \lambda_2]$ for some fixed

$s \in (0, 1]$ and $p \geq 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \psi(u) du \right| \\ & \leq \frac{(\lambda_2 - \lambda_1)}{[216(s^2 + 3s + 2)]^{\frac{1}{q}}} \left(\frac{5}{72} \right)^{1 - \frac{1}{q}} \left\{ [(3^{-s} \times 2^{1-s} + 3s \times 2^{1-s} + 3 \times 2^{-s}) |\psi'(\lambda_2)|^q \right. \\ & \quad + (5^{s+2} \times 3^{-s} \times 2^{1-s} - 6s \times 2^{-s} - 21 \times 2^{-s} + 6s - 24) |\psi'(\lambda_1)|^q]^{\frac{1}{q}} \\ & \quad + [(3^{-s} \times 2^{1-s} + 3s \times 2^{1-s} + 3 \times 2^{-s}) |\psi'(\lambda_1)|^q \\ & \quad \left. + (5^{s+2} \times 3^{-s} \times 2^{1-s} - 6s \times 2^{-s} - 21 \times 2^{-s} + 6s - 24) |\psi'(\lambda_2)|^q]^{\frac{1}{q}} \right\}. \end{aligned}$$

In [11], Sarikaya et al. gave the following results Simpson type inequalities for differentiable convex functions

Theorem 1.6. Let $\psi : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$, where $\lambda_1, \lambda_2 \in I^\circ$ with $\lambda_1 < \lambda_2$. If $|\psi'|$ is convex on $[\lambda_1, \lambda_2]$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \psi(u) du \right| \\ & \leq \frac{5(\lambda_2 - \lambda_1)}{72} [|\psi'(\lambda_1)| + |\psi'(\lambda_2)|]. \end{aligned}$$

Theorem 1.7. Let $\psi : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$, where $\lambda_1, \lambda_2 \in I^\circ$ with $\lambda_1 < \lambda_2$. If $|\psi'|^q$ is convex on $[\lambda_1, \lambda_2]$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \psi(u) du \right| \\ & \leq \frac{\lambda_2 - \lambda_1}{12} \left(\frac{1 + 2^{1+p}}{3(1+p)} \right)^{\frac{1}{p}} \left(\left(\frac{3|\psi'(\lambda_1)|^q + |\psi'(\lambda_2)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{|\psi'(\lambda_1)|^q + 3|\psi'(\lambda_2)|^q}{4} \right)^{\frac{1}{q}} \right). \end{aligned}$$

Theorem 1.8. Let $\psi : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$, where $\lambda_1, \lambda_2 \in I^\circ$ with $\lambda_1 < \lambda_2$. If $|\psi'|^q$ is convex on $[\lambda_1, \lambda_2]$, $q \geq 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \psi(u) du \right| \\ & \leq \frac{\lambda_2 - \lambda_1}{72} 5^{1 - \frac{1}{q}} \left(\left(\frac{61|\psi'(\lambda_1)|^q + 29|\psi'(\lambda_2)|^q}{18} \right)^{\frac{1}{q}} + \left(\frac{29|\psi'(\lambda_1)|^q + 61|\psi'(\lambda_2)|^q}{15} \right)^{\frac{1}{q}} \right). \end{aligned}$$

In [12], Sarikaya et al. generalized the results given in [11] for differentiable s -convex functions.

Theorem 1.9. Let $\psi : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$, where $\lambda_1, \lambda_2 \in I^\circ$ with $\lambda_1 < \lambda_2$. If $|\psi'|^q$ is s -convex on $[\lambda_1, \lambda_2]$ for some fixed $s \in (0, 1]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \psi(u) du \right| \\ & \leq \frac{\lambda_2 - \lambda_1}{12} \left(\frac{1 + 2^{1+p}}{3(1+p)} \right)^{\frac{1}{p}} \\ & \quad \times \left(\left(\frac{|\psi'(\lambda_1)|^q + \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q}{s+1} \right)^{\frac{1}{q}} + \left(\frac{\left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q + |\psi'(\lambda_2)|^q}{s+1} \right)^{\frac{1}{q}} \right). \end{aligned}$$

Theorem 1.10. Let $\psi : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$, where $\lambda_1, \lambda_2 \in I^\circ$ with $\lambda_1 < \lambda_2$. If $|\psi'|^q$ is s -convex on $[\lambda_1, \lambda_2]$ for some fixed $s \in (0, 1]$ and $q \geq 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \psi(u) du \right| \\ & \leq \frac{\lambda_2 - \lambda_1}{2} \left(\frac{5}{36} \right)^{1 - \frac{1}{q}} \\ & \quad \times \left(\left(\frac{(2s+1)3^{s+1} + 2}{3 \times 6^{s+1}(s+1)(s+2)} |\psi'(\lambda_1)|^q + \frac{2 \times 5^{s+2} + 6^{s+1}(s-4) - 3^{s+1}(2s+7)}{3 \times 6^{s+1}(s+1)(s+2)} |\psi'(\lambda_2)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{2 \times 5^{s+2} + 6^{s+1}(s-4) - 3^{s+1}(2s+7)}{3 \times 6^{s+1}(s+1)(s+2)} |\psi'(\lambda_1)|^q + \frac{(2s+1)3^{s+1} + 2}{3 \times 6^{s+1}(s+1)(s+2)} |\psi'(\lambda_2)|^q \right)^{\frac{1}{q}} \right). \end{aligned}$$

In [13], Matloka gave the following weighted Simpson type inequalities.

Theorem 1.11. Let $\psi : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be a differentiable mapping on (λ_1, λ_2) such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$ with $\lambda_1 < \lambda_2$ and $w : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be continuous and symmetric to $\frac{\lambda_1 + \lambda_2}{2}$. If $|\psi'|$ is s -convex on $[\lambda_1, \lambda_2]$ for some fixed $s \in (0, 1]$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\ & \leq \frac{(\lambda_2 - \lambda_1)^2}{1+s} \|w\|_{[\lambda_1, \lambda_2], \infty} (|\psi'(\lambda_1)| + |\psi'(\lambda_2)|). \end{aligned}$$

Theorem 1.12. Let $\psi : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be a differentiable mapping on (λ_1, λ_2) such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$ with $\lambda_1 < \lambda_2$ and $w : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be continuous and symmetric to $\frac{\lambda_1 + \lambda_2}{2}$. If $|\psi'|^q$ is s -convex on $[\lambda_1, \lambda_2]$ for some fixed $s \in (0, 1]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\ & \leq \frac{(\lambda_2 - \lambda_1)^2}{12} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(\frac{1 + 2^{1+p}}{3(1+p)} \right)^{\frac{1}{p}} \left(\frac{1}{4} \right)^{\frac{1}{q}} \left(\frac{2}{s+1} \right)^{\frac{1}{q}} \\ & \quad \times \left\{ \left(\left(\frac{1}{2} \right)^{s+1} |\psi'(\lambda_1)|^q + \left[1 - \left(\frac{1}{2} \right)^{s+1} \right] |\psi'(\lambda_2)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\left[1 - \left(\frac{1}{2} \right)^{s+1} \right] |\psi'(\lambda_1)|^q + \left(\frac{1}{2} \right)^{s+1} |\psi'(\lambda_2)|^q \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

In this paper, inspired by the above mentioned studies, we will establish a new identity and then applying it to derive new weighted Simpson type inequalities for s -convex functions in the second sense. In order to illustrate the efficient of our main results, some applications to special means and weighted Simpson quadrature formula will be obtain. At the end, a brief conclusion will be provided as well.

2. Main results

Lemma 2.1. Let $\psi : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , $\lambda_1, \lambda_2 \in I^\circ$ with $\lambda_1 < \lambda_2$, and let $w : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be symmetric with respect to $\frac{\lambda_1 + \lambda_2}{2}$. If $\psi', w \in \mathcal{L}[\lambda_1, \lambda_2]$, then

$$\begin{aligned} & \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du = \frac{3(\lambda_2 - \lambda_1)^2}{4} \\ & \quad \times \left[\int_0^1 p_1(\chi) \psi' \left(\chi \lambda_1 + (1 - \chi) \frac{\lambda_1 + \lambda_2}{2} \right) d\chi + \int_0^1 p_2(\chi) \psi' \left(\chi \frac{\lambda_1 + \lambda_2}{2} + (1 - \chi) \lambda_2 \right) d\chi \right], \end{aligned}$$

where

$$p_1(\chi) = \frac{2}{9} \int_0^1 w \left(\zeta \lambda_1 + (1 - \zeta) \frac{\lambda_1 + \lambda_2}{2} \right) d\zeta - \frac{1}{3} \int_0^\chi w \left(\zeta \lambda_1 + (1 - \zeta) \frac{\lambda_1 + \lambda_2}{2} \right) d\zeta \quad (2.1)$$

and

$$p_2(\chi) = \frac{1}{9} \int_0^1 w \left(\zeta \frac{\lambda_1 + \lambda_2}{2} + (1 - \zeta) \lambda_2 \right) d\zeta - \frac{1}{3} \int_0^\chi w \left(\zeta \frac{\lambda_1 + \lambda_2}{2} + (1 - \zeta) \lambda_2 \right) d\zeta. \quad (2.2)$$

Proof. Integrating by parts and changing the variables, and using the symmetry of w , we obtain

$$\begin{aligned}
& \int_0^1 p_1(\chi) \psi' \left(\chi \lambda_1 + (1-\chi) \frac{\lambda_1 + \lambda_2}{2} \right) d\chi \\
&= \int_0^1 \left(\frac{2}{9} \int_0^1 w \left(\zeta \lambda_1 + (1-\zeta) \frac{\lambda_1 + \lambda_2}{2} \right) d\zeta - \frac{1}{3} \int_0^{\chi} w \left(\zeta \lambda_1 + (1-\zeta) \frac{\lambda_1 + \lambda_2}{2} \right) d\zeta \right) \\
&\quad \times \psi' \left(\chi \lambda_1 + (1-\chi) \frac{\lambda_1 + \lambda_2}{2} \right) d\chi \\
&= \frac{2}{\lambda_2 - \lambda_1} \left(\frac{2}{9} \int_0^1 w \left(\zeta \lambda_1 + (1-\zeta) \frac{\lambda_1 + \lambda_2}{2} \right) d\zeta - \frac{1}{3} \int_0^{\chi} w \left(\zeta \lambda_1 + (1-\zeta) \frac{\lambda_1 + \lambda_2}{2} \right) d\zeta \right) \\
&\quad \times \psi \left(\chi \lambda_1 + (1-\chi) \frac{\lambda_1 + \lambda_2}{2} \right) \Big|_{\chi=0}^{\chi=1} \\
&\quad - \frac{2}{3(\lambda_2 - \lambda_1)} \int_0^1 w \left(\chi \lambda_1 + (1-\chi) \frac{\lambda_1 + \lambda_2}{2} \right) \psi \left(\chi \lambda_1 + (1-\chi) \frac{\lambda_1 + \lambda_2}{2} \right) d\chi \\
&= \frac{2}{9(\lambda_2 - \lambda_1)} \left(\int_0^1 w \left(\zeta \lambda_1 + (1-\zeta) \frac{\lambda_1 + \lambda_2}{2} \right) d\zeta \right) \psi(\lambda_1) \\
&\quad + \frac{4}{9(\lambda_2 - \lambda_1)} \left(\int_0^1 w \left(\zeta \lambda_1 + (1-\zeta) \frac{\lambda_1 + \lambda_2}{2} \right) d\zeta \right) \psi \left(\frac{\lambda_1 + \lambda_2}{2} \right) \\
&\quad - \frac{2}{3(\lambda_2 - \lambda_1)} \int_0^1 w \left(\chi \lambda_1 + (1-\chi) \frac{\lambda_1 + \lambda_2}{2} \right) \psi \left(\chi \lambda_1 + (1-\chi) \frac{\lambda_1 + \lambda_2}{2} \right) d\chi \\
&= \frac{4}{9(\lambda_2 - \lambda_1)^2} \left(\int_{\lambda_1}^{\frac{\lambda_1 + \lambda_2}{2}} w(u) du \right) \psi(\lambda_1) + \frac{8}{9(\lambda_2 - \lambda_1)^2} \left(\int_{\lambda_1}^{\frac{\lambda_1 + \lambda_2}{2}} w(u) du \right) \psi \left(\frac{\lambda_1 + \lambda_2}{2} \right) \\
&\quad - \frac{4}{3(\lambda_2 - \lambda_1)^2} \int_{\lambda_1}^{\frac{\lambda_1 + \lambda_2}{2}} w(u) \psi(u) du
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{9(\lambda_2 - \lambda_1)^2} \left(\int_{\lambda_1}^{\lambda_2} w(u) \, du \right) \psi(\lambda_1) + \frac{4}{9(\lambda_2 - \lambda_1)^2} \left(\int_{\lambda_1}^{\lambda_2} w(u) \, du \right) \psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) \\
&\quad - \frac{4}{3(\lambda_2 - \lambda_1)^2} \int_{\lambda_1}^{\frac{\lambda_1 + \lambda_2}{2}} w(u) \psi(u) \, du.
\end{aligned} \tag{2.3}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1 p_2(\chi) \psi'(\chi\lambda_1 + (1-\chi)\lambda_2) \, d\chi \\
&= \int_0^1 \left(\frac{1}{9} \int_0^1 w\left(\zeta\frac{\lambda_1 + \lambda_2}{2} + (1-\zeta)\lambda_2\right) \, d\zeta - \frac{1}{3} \int_0^\chi w\left(\zeta\frac{\lambda_1 + \lambda_2}{2} + (1-\zeta)\lambda_2\right) \, d\zeta \right) \\
&\quad \times \psi'\left(\chi\frac{\lambda_1 + \lambda_2}{2} + (1-\chi)\lambda_2\right) \, d\chi \\
&= -\frac{2}{\lambda_2 - \lambda_1} \left(\frac{1}{9} \int_0^1 w\left(\zeta\frac{\lambda_1 + \lambda_2}{2} + (1-\zeta)\lambda_2\right) \, d\zeta - \frac{1}{3} \int_0^\chi w\left(\zeta\frac{\lambda_1 + \lambda_2}{2} + (1-\zeta)\lambda_2\right) \, d\zeta \right) \\
&\quad \times \psi\left(\chi\frac{\lambda_1 + \lambda_2}{2} + (1-\chi)\lambda_2\right) \Bigg|_{\chi=0}^{\chi=1} \\
&\quad - \frac{2}{3(\lambda_2 - \lambda_1)} \int_0^1 w\left(\chi\frac{\lambda_1 + \lambda_2}{2} + (1-\chi)\lambda_2\right) \psi\left(\chi\frac{\lambda_1 + \lambda_2}{2} + (1-\chi)\lambda_2\right) \, d\chi \\
&= \frac{4}{9(\lambda_2 - \lambda_1)} \left(\int_0^1 w\left(\zeta\frac{\lambda_1 + \lambda_2}{2} + (1-\zeta)\lambda_2\right) \, d\zeta \right) \psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) \\
&\quad + \frac{2}{9(\lambda_2 - \lambda_1)} \left(\int_0^1 w\left(\zeta\frac{\lambda_1 + \lambda_2}{2} + (1-\zeta)\lambda_2\right) \, d\zeta \right) \psi(\lambda_2) \\
&\quad - \frac{2}{3(\lambda_2 - \lambda_1)} \int_0^1 w\left(\chi\frac{\lambda_1 + \lambda_2}{2} + (1-\chi)\lambda_2\right) \psi\left(\chi\frac{\lambda_1 + \lambda_2}{2} + (1-\chi)\lambda_2\right) \, d\chi \\
&= \frac{8}{9(\lambda_2 - \lambda_1)^2} \left(\int_{\frac{\lambda_1 + \lambda_2}{2}}^{\lambda_2} w(u) \, du \right) \psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \frac{4}{9(\lambda_2 - \lambda_1)^2} \left(\int_{\frac{\lambda_1 + \lambda_2}{2}}^{\lambda_2} w(u) \, du \right) \psi(\lambda_2)
\end{aligned}$$

$$\begin{aligned}
& - \frac{4}{3(\lambda_2 - \lambda_1)^2} \int_{\frac{\lambda_1 + \lambda_2}{2}}^{\lambda_2} w(u) \psi(u) du \\
& = \frac{4}{9(\lambda_2 - \lambda_1)^2} \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) \psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \frac{2}{9(\lambda_2 - \lambda_1)^2} \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) \psi(\lambda_2) \\
& - \frac{4}{3(\lambda_2 - \lambda_1)^2} \int_{\frac{\lambda_1 + \lambda_2}{2}}^{\lambda_2} w(u) \psi(u) du. \tag{2.4}
\end{aligned}$$

Summing (2.3) and (2.4), and then multiplying the resulting equality by factor $\frac{3(\lambda_2 - \lambda_1)^2}{4}$, we get the desired result. $\square$

Theorem 2.2. Let $\psi : [\lambda_1, \lambda_2] \rightarrow \mathfrak{R}$ be a differentiable function on (λ_1, λ_2) such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$ with $0 \leq \lambda_1 < \lambda_2$, and let $w : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be continuous and symmetric function with respect to $\frac{\lambda_1 + \lambda_2}{2}$. If $|\psi'|$ is s -convex in the second sense for some fixed $s \in (0, 1]$, then

$$\begin{aligned}
& \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\
& \leq \frac{3(\lambda_2 - \lambda_1)^2}{4(1+s)(2+s)} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(\left(\frac{s-1}{9} + \left(\frac{2}{3}\right)^{3+s} \right) |\psi'(\lambda_1)| \right. \\
& \quad \left. + 2 \left(\frac{1+2s}{9} + 2 \left(\frac{1}{3}\right)^{3+s} \right) \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| + \left(\frac{s-1}{9} + \left(\frac{2}{3}\right)^{3+s} \right) |\psi'(\lambda_2)| \right).
\end{aligned}$$

Proof. From Lemma 2.1, and properties of modulus, we have

$$\begin{aligned}
& \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\
& \leq \frac{3(\lambda_2 - \lambda_1)^2}{4} \int_0^1 |p_1(\chi)| \left| \psi'\left(\chi\lambda_1 + (1-\chi)\frac{\lambda_1 + \lambda_2}{2}\right) \right| d\chi \\
& \quad + \frac{3(\lambda_2 - \lambda_1)^2}{4} \int_0^1 |p_2(\chi)| \left| \psi'\left(\chi\frac{\lambda_1 + \lambda_2}{2} + (1-\chi)\lambda_2\right) \right| d\chi. \tag{2.5}
\end{aligned}$$

Since $|\psi'|$ is s -convex, we get

$$\left| \psi'\left(\chi\lambda_1 + (1-\chi)\frac{\lambda_1 + \lambda_2}{2}\right) \right| \leq \chi^s |\psi'(\lambda_1)| + (1-\chi)^s \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right|, \tag{2.6}$$

$$\left| \psi'\left(\chi\frac{\lambda_1 + \lambda_2}{2} + (1-\chi)\lambda_2\right) \right| \leq \chi^s \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| + (1-\chi)^s |\psi'(\lambda_2)|. \tag{2.7}$$

Using (2.1), (2.2), (2.6) and (2.7) in (2.5), we obtain

$$\begin{aligned}
& \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\
& \leq \frac{3(\lambda_2 - \lambda_1)^2}{4} \int_0^1 \left| \frac{2}{9} \int_0^1 w\left(\zeta\lambda_1 + (1-\zeta)\frac{\lambda_1 + \lambda_2}{2}\right) d\zeta - \frac{1}{3} \int_0^{\chi} w\left(\zeta\lambda_1 + (1-\zeta)\frac{\lambda_1 + \lambda_2}{2}\right) d\zeta \right| \\
& \quad \times \left(\chi^s |\psi'(\lambda_1)| + (1-\chi)^s \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| \right) d\chi \\
& \quad + \frac{3(\lambda_2 - \lambda_1)^2}{4} \int_0^1 \left| \frac{1}{9} \int_0^1 w\left(\zeta\frac{\lambda_1 + \lambda_2}{2} + (1-\zeta)\lambda_2\right) d\zeta - \frac{1}{3} \int_0^{\chi} w\left(\zeta\frac{\lambda_1 + \lambda_2}{2} + (1-\zeta)\lambda_2\right) d\zeta \right| \\
& \quad \times \left(\chi^s \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| + (1-\chi)^s |\psi'(\lambda_2)| \right) d\chi \\
& \leq \frac{3(\lambda_2 - \lambda_1)^2}{4} \|w\|_{[\lambda_1, \lambda_2], \infty} \int_0^1 \left| \frac{2}{9} \int_0^1 d\zeta - \frac{1}{3} \int_0^{\chi} d\zeta \right| \left(\chi^s |\psi'(\lambda_1)| + (1-\chi)^s \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| \right) d\chi \\
& \quad + \frac{3(\lambda_2 - \lambda_1)^2}{4} \|w\|_{[\lambda_1, \lambda_2], \infty} \int_0^1 \left| \frac{1}{9} \int_0^1 d\zeta - \frac{1}{3} \int_0^{\chi} d\zeta \right| \left(\chi^s \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| + (1-\chi)^s |\psi'(\lambda_2)| \right) d\chi \\
& = \frac{3(\lambda_2 - \lambda_1)^2}{4} \|w\|_{[\lambda_1, \lambda_2], \infty} \int_0^1 \left| \frac{2}{9} - \frac{1}{3}\chi \right| \left(\chi^s |\psi'(\lambda_1)| + (1-\chi)^s \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| \right) d\chi \\
& \quad + \frac{3(\lambda_2 - \lambda_1)^2}{4} \|w\|_{[\lambda_1, \lambda_2], \infty} \int_0^1 \left| \frac{1}{9} - \frac{1}{3}\chi \right| \left(\chi^s \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| + (1-\chi)^s |\psi'(\lambda_2)| \right) d\chi \\
& = \frac{3(\lambda_2 - \lambda_1)^2}{4} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(|\psi'(\lambda_1)| \int_0^1 \left| \frac{2}{9} - \frac{1}{3}\chi \right| \chi^s d\chi + \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| \int_0^1 \left| \frac{2}{9} - \frac{1}{3}\chi \right| (1-\chi)^s d\chi \right. \\
& \quad \left. + \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| \int_0^1 \left| \frac{1}{9} - \frac{1}{3}\chi \right| \chi^s d\chi + |\psi'(\lambda_2)| \int_0^1 \left| \frac{1}{9} - \frac{1}{3}\chi \right| (1-\chi)^s d\chi \right) \\
& = \frac{3(\lambda_2 - \lambda_1)^2}{4(1+s)(2+s)} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(\left(\frac{s-1}{9} + \left(\frac{2}{3}\right)^{3+s} \right) |\psi'(\lambda_1)| \right. \\
& \quad \left. + 2 \left(\frac{1+2s}{9} + 2 \left(\frac{1}{3}\right)^{3+s} \right) \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| + \left(\frac{s-1}{9} + \left(\frac{2}{3}\right)^{3+s} \right) |\psi'(\lambda_2)| \right),
\end{aligned}$$

where we have used the fact that

$$\begin{aligned} \int_0^1 \left| \frac{2}{9} - \frac{1}{3}\chi \right| \chi^s d\chi &= \int_0^1 \left| \frac{1}{9} - \frac{1}{3}\chi \right| (1-\chi)^s d\chi \\ &= \frac{1}{(1+s)(2+s)} \left(\frac{s-1}{9} + \left(\frac{2}{3} \right)^{3+s} \right) \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} \int_0^1 \left| \frac{1}{9} - \frac{1}{3}\chi \right| \chi^s d\chi &= \int_0^1 \left| \frac{2}{9} - \frac{1}{3}\chi \right| (1-\chi)^s d\chi \\ &= \frac{1}{(1+s)(2+s)} \left(\frac{1+2s}{9} + 2 \left(\frac{1}{3} \right)^{3+s} \right). \end{aligned} \quad (2.9)$$

The proof is completed. $\square$

Corollary 2.3. In Theorem 2.2, if we take $w(u) = \frac{1}{\lambda_2 - \lambda_1}$, we have

$$\begin{aligned} &\left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \psi(u) du \right| \\ &\leq \frac{3(\lambda_2 - \lambda_1)}{4(1+s)(2+s)} \left(\left(\frac{s-1}{9} + \left(\frac{2}{3} \right)^{3+s} \right) |\psi'(\lambda_1)| \right. \\ &\quad \left. + 2 \left(\frac{1+2s}{9} + 2 \left(\frac{1}{3} \right)^{3+s} \right) \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| + \left(\frac{s-1}{9} + \left(\frac{2}{3} \right)^{3+s} \right) |\psi'(\lambda_2)| \right). \end{aligned}$$

Corollary 2.4. In Theorem 2.2, if we take $s = 1$, we get

$$\begin{aligned} &\left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\ &\leq \frac{(\lambda_2 - \lambda_1)^2}{324} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(8 |\psi'(\lambda_1)| + 29 \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| + 8 |\psi'(\lambda_2)| \right). \end{aligned}$$

Moreover, if we choose $w(u) = \frac{1}{\lambda_2 - \lambda_1}$, we obtain

$$\begin{aligned} &\left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \psi(u) du \right| \\ &\leq \frac{\lambda_2 - \lambda_1}{324} \left(8 |\psi'(\lambda_1)| + 29 \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right| + 8 |\psi'(\lambda_2)| \right). \end{aligned}$$

Corollary 2.5. In Corollary 2.4, using the convexity of $|\psi'|$, we have

$$\left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\ \leq \frac{5(\lambda_2 - \lambda_1)^2}{72} \|w\|_{[\lambda_1, \lambda_2], \infty} (|\psi'(\lambda_1)| + |\psi'(\lambda_2)|).$$

Moreover, if we choose $w(u) = \frac{1}{\lambda_2 - \lambda_1}$, we obtain Theorem 1.6 from [11], also Corollary 1 from [10].

Theorem 2.6. Let $\psi : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be a differentiable function on (λ_1, λ_2) such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$ with $0 \leq \lambda_1 < \lambda_2$, and let $w : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be continuous and symmetric function with respect to $\frac{\lambda_1 + \lambda_2}{2}$. If $|\psi'|^q$ is s -convex in the second sense for some fixed $s \in (0, 1]$, where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\ \leq \frac{(\lambda_2 - \lambda_1)^2}{12} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(\frac{1 + 2^{1+p}}{3(1+p)} \right)^{\frac{1}{p}} \left(\frac{1}{1+s} \right)^{\frac{1}{q}} \\ \times \left(\left(|\psi'(\lambda_1)|^q + \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q \right)^{\frac{1}{q}} + \left(\left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q + |\psi'(\lambda_2)|^q \right)^{\frac{1}{q}} \right).$$

Proof. From Lemma 2.1, properties of modulus, Hölder inequality, and s -convexity of $|\psi'|^q$, we have

$$\left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\ \leq \frac{3(\lambda_2 - \lambda_1)^2}{4} \left(\int_0^1 |p_1(\chi)|^p d\chi \right)^{\frac{1}{p}} \left(\int_0^1 \left| \psi'\left(\chi\lambda_1 + (1-\chi)\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q d\chi \right)^{\frac{1}{q}} \\ + \frac{3(\lambda_2 - \lambda_1)^2}{4} \left(\int_0^1 |p_2(\chi)|^p d\chi \right)^{\frac{1}{p}} \left(\int_0^1 \left| \psi'\left(\chi\frac{\lambda_1 + \lambda_2}{2} + (1-\chi)\lambda_2\right) \right|^q d\chi \right)^{\frac{1}{q}} \\ \leq \frac{3(\lambda_2 - \lambda_1)^2}{4} \left(\int_0^1 \left| \frac{2}{9} \int_0^1 w\left(\zeta\lambda_1 + (1-\zeta)\frac{\lambda_1 + \lambda_2}{2}\right) d\zeta - \frac{1}{3} \int_0^{\chi} w\left(\zeta\lambda_1 + (1-\zeta)\frac{\lambda_1 + \lambda_2}{2}\right) d\zeta \right|^p d\chi \right)^{\frac{1}{p}}$$

$$\begin{aligned}
& \times \left(\int_0^1 \left(\chi^s |\psi'(\lambda_1)|^q + (1-\chi)^s \left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q \right) d\chi \right)^{\frac{1}{q}} \\
& + \frac{3(\lambda_2 - \lambda_1)^2}{4} \left(\int_0^1 \left| \frac{1}{9} \int_0^1 w \left(\zeta \frac{\lambda_1 + \lambda_2}{2} + (1-\zeta)\lambda_2 \right) d\zeta - \frac{1}{3} \int_0^x w \left(\zeta \frac{\lambda_1 + \lambda_2}{2} + (1-\zeta)\lambda_2 \right) d\zeta \right|^p d\chi \right)^{\frac{1}{p}} \\
& \times \left(\int_0^1 \left(\chi^s \left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q + (1-\chi)^s |\psi'(\lambda_2)|^q \right) d\chi \right)^{\frac{1}{q}} \\
& \leq \frac{3(\lambda_2 - \lambda_1)^2}{4(1+s)^{\frac{1}{q}}} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(\int_0^1 \left| \frac{2}{9} - \frac{1}{3}\chi \right|^p d\chi \right)^{\frac{1}{p}} \left(|\psi'(\lambda_1)|^q + \left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q \right)^{\frac{1}{q}} \\
& + \frac{3(\lambda_2 - \lambda_1)^2}{4(1+s)^{\frac{1}{q}}} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(\int_0^1 \left| \frac{1}{9} - \frac{1}{3}\chi \right|^p d\chi \right)^{\frac{1}{p}} \left(\left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q + |\psi'(\lambda_2)|^q \right)^{\frac{1}{q}} \\
& = \frac{(\lambda_2 - \lambda_1)^2}{12} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(\frac{1+2^{1+p}}{3(1+p)} \right)^{\frac{1}{p}} \left(\frac{1}{1+s} \right)^{\frac{1}{q}} \\
& \times \left(\left(|\psi'(\lambda_1)|^q + \left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q \right)^{\frac{1}{q}} + \left(\left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q + |\psi'(\lambda_2)|^q \right)^{\frac{1}{q}} \right).
\end{aligned}$$

The proof is completed. $\square$

Remark 2.7. In Theorem 2.6, if we take $w(u) = \frac{1}{\lambda_2 - \lambda_1}$, we obtain Theorem 1.9 from [12].

Corollary 2.8. In Theorem 2.6, if we take $s = 1$, we have

$$\begin{aligned}
& \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi \left(\frac{\lambda_1 + \lambda_2}{2} \right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\
& \leq \frac{(\lambda_2 - \lambda_1)^2}{12} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(\frac{1+2^{1+p}}{3(1+p)} \right)^{\frac{1}{p}} \left(\frac{1}{2} \right)^{\frac{1}{q}} \\
& \left(\left(|\psi'(\lambda_1)|^q + \left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q \right)^{\frac{1}{q}} + \left(\left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q + |\psi'(\lambda_2)|^q \right)^{\frac{1}{q}} \right).
\end{aligned}$$

Moreover, if we choose $w(u) = \frac{1}{\lambda_2 - \lambda_1}$, we get Corollary 3 from [12].

Corollary 2.9. In Corollary 2.8, using the convexity of $|\psi'|^q$, we obtain

$$\left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\ \leq \frac{\lambda_2 - \lambda_1}{12} \left(\frac{1 + 2^{1+p}}{3(1+p)} \right)^{\frac{1}{p}} \left(\left(\frac{3|\psi'(\lambda_1)|^q + |\psi'(\lambda_2)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{|\psi'(\lambda_1)|^q + 3|\psi'(\lambda_2)|^q}{4} \right)^{\frac{1}{q}} \right).$$

Moreover, if we choose $w(u) = \frac{1}{\lambda_2 - \lambda_1}$, we have Theorem 1.7 from [11].

Theorem 2.10. Let $\psi : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be a differentiable function on (λ_1, λ_2) such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$ with $0 \leq \lambda_1 < \lambda_2$, and let $w : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be continuous and symmetric function with respect to $\frac{\lambda_1 + \lambda_2}{2}$. If $|\psi'|^q$ is s -convex in the second sense for some fixed $s \in (0, 1]$ and $q \geq 1$, then

$$\left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\ \leq \frac{3(\lambda_2 - \lambda_1)^2}{4} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(\frac{5}{54} \right)^{1 - \frac{1}{q}} \left(\frac{1}{(1+s)(2+s)} \right)^{\frac{1}{q}} \\ \times \left(\left(\left(\frac{s-1}{9} + \left(\frac{2}{3} \right)^{3+s} \right) |\psi'(\lambda_1)|^q + \left(\frac{1+2s}{9} + 2 \left(\frac{1}{3} \right)^{3+s} \right) \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ \left. + \left(\left(\frac{1+2s}{9} + 2 \left(\frac{1}{3} \right)^{3+s} \right) \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q + \left(\frac{s-1}{9} + \left(\frac{2}{3} \right)^{3+s} \right) |\psi'(\lambda_2)|^q \right)^{\frac{1}{q}} \right).$$

Proof. From Lemma 2.1, properties of modulus, power mean inequality, s -convexity of $|\psi'|^q$, (2.8) and (2.9), we have

$$\left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\ \leq \frac{3(\lambda_2 - \lambda_1)^2}{4} \left(\int_0^1 |p_1(\chi)| d\chi \right)^{1 - \frac{1}{q}} \left(\int_0^1 |p_1(\chi)| \left| \psi'\left(\chi\lambda_1 + (1-\chi)\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q d\chi \right)^{\frac{1}{q}} \\ + \frac{3(\lambda_2 - \lambda_1)^2}{4} \left(\int_0^1 |p_2(\chi)| d\chi \right)^{1 - \frac{1}{q}} \left(\int_0^1 |p_2(\chi)| \left| \psi'\left(\chi\frac{\lambda_1 + \lambda_2}{2} + (1-\chi)\lambda_2\right) \right|^q d\chi \right)^{\frac{1}{q}} \\ \leq \frac{3(\lambda_2 - \lambda_1)^2}{4} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(\int_0^1 \left| \frac{2}{9} - \frac{1}{3}\chi \right| d\chi \right)^{1 - \frac{1}{q}}$$

$$\begin{aligned}
& \times \left(\int_0^1 \left| \frac{2}{9} - \frac{1}{3}\chi \right| \left(\chi^s |\psi'(\lambda_1)|^q + (1-\chi)^s \left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q \right) d\chi \right)^{\frac{1}{q}} \\
& + \frac{3(\lambda_2 - \lambda_1)^2}{4} \|\mathbf{w}\|_{[\lambda_1, \lambda_2], \infty} \left(\int_0^1 \left| \frac{1}{9} - \frac{1}{3}\chi \right| d\chi \right)^{1-\frac{1}{q}} \\
& \times \left(\int_0^1 \left| \frac{1}{9} - \frac{1}{3}\chi \right| \left(\chi^s \left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q + (1-\chi)^s |\psi'(\lambda_2)|^q \right) d\chi \right)^{\frac{1}{q}} \\
& = \frac{3(\lambda_2 - \lambda_1)^2}{4} \|\mathbf{w}\|_{[\lambda_1, \lambda_2], \infty} \left(\int_0^1 \left| \frac{2}{9} - \frac{1}{3}\chi \right| d\chi \right)^{1-\frac{1}{q}} \\
& \times \left(|\psi'(\lambda_1)|^q \int_0^1 \left| \frac{2}{9} - \frac{1}{3}\chi \right| \chi^s d\chi + \left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q \int_0^1 \left| \frac{2}{9} - \frac{1}{3}\chi \right| (1-\chi)^s d\chi \right)^{\frac{1}{q}} \\
& + \frac{3(\lambda_2 - \lambda_1)^2}{4} \|\mathbf{w}\|_{[\lambda_1, \lambda_2], \infty} \left(\int_0^1 \left| \frac{1}{9} - \frac{1}{3}\chi \right| d\chi \right)^{1-\frac{1}{q}} \\
& \times \left(\left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q \int_0^1 \left| \frac{1}{9} - \frac{1}{3}\chi \right| \chi^s d\chi + |\psi'(\lambda_2)|^q \int_0^1 \left| \frac{1}{9} - \frac{1}{3}\chi \right| (1-\chi)^s d\chi \right)^{\frac{1}{q}} \\
& = \frac{3(\lambda_2 - \lambda_1)^2}{4} \|\mathbf{w}\|_{[\lambda_1, \lambda_2], \infty} \left(\frac{5}{54} \right)^{1-\frac{1}{q}} \left(\frac{1}{(1+s)(2+s)} \right)^{\frac{1}{q}} \\
& \times \left(\left(\left(\frac{s-1}{9} + \left(\frac{2}{3} \right)^{3+s} \right) |\psi'(\lambda_1)|^q + \left(\frac{1+2s}{9} + 2 \left(\frac{1}{3} \right)^{3+s} \right) \left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q \right)^{\frac{1}{q}} \right. \\
& \left. + \left(\left(\frac{1+2s}{9} + 2 \left(\frac{1}{3} \right)^{3+s} \right) \left| \psi' \left(\frac{\lambda_1 + \lambda_2}{2} \right) \right|^q + \left(\frac{s-1}{9} + \left(\frac{2}{3} \right)^{3+s} \right) |\psi'(\lambda_2)|^q \right)^{\frac{1}{q}} \right).
\end{aligned}$$

The proof is completed. $\square$

Corollary 2.11. In Theorem 2.10, if we take $w(u) = \frac{1}{\lambda_2 - \lambda_1}$, we get

$$\begin{aligned} & \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \psi(u) du \right| \\ & \leq \frac{3(\lambda_2 - \lambda_1)}{4} \left(\frac{5}{54} \right)^{1 - \frac{1}{q}} \left(\frac{1}{(1+s)(2+s)} \right)^{\frac{1}{q}} \\ & \quad \times \left(\left(\left(\frac{s-1}{9} + \left(\frac{2}{3} \right)^{3+s} \right) |\psi'(\lambda_1)|^q + \left(\frac{1+2s}{9} + 2 \left(\frac{1}{3} \right)^{3+s} \right) \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\left(\frac{1+2s}{9} + 2 \left(\frac{1}{3} \right)^{3+s} \right) \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q + \left(\frac{s-1}{9} + \left(\frac{2}{3} \right)^{3+s} \right) |\psi'(\lambda_2)|^q \right)^{\frac{1}{q}} \right). \end{aligned}$$

Corollary 2.12. In Theorem 2.10, if we take $s = 1$, we obtain

$$\begin{aligned} & \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\ & \leq \frac{5(\lambda_2 - \lambda_1)^2}{72} \|w\|_{[\lambda_1, \lambda_2], \infty} \\ & \quad \times \left(\left(\frac{16|\psi'(\lambda_1)|^q + 29 \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q}{45} \right)^{\frac{1}{q}} + \left(\frac{29 \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q + 16|\psi'(\lambda_2)|^q}{45} \right)^{\frac{1}{q}} \right). \end{aligned}$$

Moreover, if we choose $w(u) = \frac{1}{\lambda_2 - \lambda_1}$, we have

$$\begin{aligned} & \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \psi(u) du \right| \\ & \leq \frac{5(\lambda_2 - \lambda_1)}{72} \left(\left(\frac{16|\psi'(\lambda_1)|^q + 29 \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q}{45} \right)^{\frac{1}{q}} + \left(\frac{29 \left| \psi'\left(\frac{\lambda_1 + \lambda_2}{2}\right) \right|^q + 16|\psi'(\lambda_2)|^q}{45} \right)^{\frac{1}{q}} \right). \end{aligned}$$

Corollary 2.13. In Corollary 2.12, using the convexity of $|\psi'|^q$, we get

$$\begin{aligned} & \left| \frac{1}{6} \left(\psi(\lambda_1) + 4\psi\left(\frac{\lambda_1 + \lambda_2}{2}\right) + \psi(\lambda_2) \right) \left(\int_{\lambda_1}^{\lambda_2} w(u) du \right) - \int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du \right| \\ & \leq \frac{5(\lambda_2 - \lambda_1)^2}{72} \|w\|_{[\lambda_1, \lambda_2], \infty} \\ & \quad \times \left(\left(\frac{61|\psi'(\lambda_1)|^q + 29|\psi'(\lambda_2)|^q}{90} \right)^{\frac{1}{q}} + \left(\frac{29|\psi'(\lambda_1)|^q + 61|\psi'(\lambda_2)|^q}{90} \right)^{\frac{1}{q}} \right). \end{aligned}$$

Moreover, if we choose $w(u) = \frac{1}{\lambda_2 - \lambda_1}$, we obtain Theorem 1.8 from [11].

3. Applications

3.1. Special means

We shall consider the following special means for different positive real numbers λ_1 and λ_2 .

- Arithmetic mean: $\mathcal{A}(\lambda_1, \lambda_2) = \frac{\lambda_1 + \lambda_2}{2}$.
- p -Logarithmic mean: $\mathcal{L}_p(\lambda_1, \lambda_2) = \left(\frac{\lambda_2^{p+1} - \lambda_1^{p+1}}{(p+1)(\lambda_2 - \lambda_1)} \right)^{\frac{1}{p}}$, $p \in \mathbb{Z} \setminus \{0, -1\}$.

Proposition 3.1. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $0 < \lambda_1 < \lambda_2$. Then

$$|\mathcal{A}(\lambda_1^3, \lambda_2^3) + 2\mathcal{A}^3(\lambda_1, \lambda_2) - 3\mathcal{L}_3^3(\lambda_1, \lambda_2)| \leq \frac{\lambda_2 - \lambda_1}{36} (16\mathcal{A}(\lambda_1^2, \lambda_2^2) + 29\mathcal{A}^2(\lambda_1, \lambda_2)).$$

Proof. The assertion follows from Theorem 2.2 with $w(u) = \frac{1}{\lambda_2 - \lambda_1}$ and $s = 1$, applied to the function $\psi(u) = u^3$. $\square$

Proposition 3.2. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $0 < \lambda_1 < \lambda_2$. Then for $q \geq 1$, we have

$$\begin{aligned} & \left| \mathcal{A}\left(\lambda_1^{\frac{1+2q}{2q}}, \lambda_2^{\frac{1+2q}{2q}}\right) + 2\mathcal{A}^{\frac{1+2q}{2q}}(\lambda_1, \lambda_2) - 3\mathcal{L}^{\frac{1+2q}{2q}}_{\frac{1+2q}{2q}}(\lambda_1, \lambda_2) \right| \\ & \leq \frac{5(\lambda_2 - \lambda_1)}{24} \frac{1+2q}{2q} \left(\left(\frac{4(16\sqrt{6}-9)\sqrt{\lambda_1} + 16(9+\sqrt{3})\sqrt{\frac{\lambda_1+\lambda_2}{2}}}{225} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{16(9+\sqrt{3})\sqrt{\frac{\lambda_1+\lambda_2}{2}} + 4(16\sqrt{6}-9)\sqrt{\lambda_2}}{225} \right)^{\frac{1}{q}} \right). \end{aligned}$$

Proof. The assertion follows from Corollary 2.11, applied to the function $\psi(u) = \frac{2q}{1+2q} u^{\frac{1+2q}{2q}}$ with $q \geq 1$, which clearly $|\psi'(u)|^q$ is $\frac{1}{2}$ -convex. $\square$

3.2. Weighted Simpson quadrature formula

Let $\mathcal{P}$ be the partition of the points $\lambda_1 = x_0 < x_1 < \dots < x_n = \lambda_2$ of the interval $[\lambda_1, \lambda_2]$ and consider the quadrature formula

$$\int_{\lambda_1}^{\lambda_2} w(u) \psi(u) du = \mathcal{S}_w(\psi, \mathcal{P}) + \mathcal{R}_w(\psi, \mathcal{P}), \quad (3.1)$$

where

$$\mathcal{S}_w(\psi, \mathcal{P}) := \frac{1}{6} \sum_{i=0}^{n-1} \left[\psi(x_i) + 4\psi\left(\frac{x_i + x_{i+1}}{2}\right) + \psi(x_{i+1}) \right] \left(\int_{x_i}^{x_{i+1}} w(u) du \right), \quad (3.2)$$

is the weighted Simpson version and $\mathcal{R}_w(\psi, \mathcal{P})$ denotes the associated approximation error. The following results are given to illustrative the implementation of above weighted Simpson quadrature formula.

Proposition 3.3. Let $n \in \mathbb{N}$ and $\psi : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be a differentiable function on (λ_1, λ_2) such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$ with $0 \leq \lambda_1 < \lambda_2$, and let $w : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be continuous and symmetric function with respect to $\frac{\lambda_1 + \lambda_2}{2}$. If $|\psi'|$ is s -convex in the second sense for some fixed $s \in (0, 1]$, then

$$\begin{aligned} |\mathcal{R}_w(\psi, \mathcal{P})| &\leq \frac{3}{4(1+s)(2+s)} \|w\|_{[\lambda_1, \lambda_2], \infty} \\ &\times \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left(\left(\frac{s-1}{9} + \left(\frac{2}{3} \right)^{3+s} \right) |\psi'(x_i)| \right. \\ &\quad \left. + 2 \left(\frac{1+2s}{9} + 2 \left(\frac{1}{3} \right)^{3+s} \right) \left| \psi' \left(\frac{x_i + x_{i+1}}{2} \right) \right| + \left(\frac{s-1}{9} + \left(\frac{2}{3} \right)^{3+s} \right) |\psi'(x_{i+1})| \right). \end{aligned}$$

Proof. Applying Theorem 2.2 on the subintervals $[x_i, x_{i+1}]$ ($i = 0, \dots, n-1$) of the partition $\mathcal{P}$, we get

$$\begin{aligned} &\left| \int_{x_i}^{x_{i+1}} w(u) \psi(u) du - \frac{1}{6} \left[\psi(x_i) + 4\psi \left(\frac{x_i + x_{i+1}}{2} \right) + \psi(x_{i+1}) \right] \left(\int_{x_i}^{x_{i+1}} w(u) du \right) \right| \\ &\leq \frac{3}{4(1+s)(2+s)} \|w\|_{[x_i, x_{i+1}], \infty} \\ &\times (x_{i+1} - x_i)^2 \left(\left(\frac{s-1}{9} + \left(\frac{2}{3} \right)^{3+s} \right) |\psi'(x_i)| \right. \\ &\quad \left. + 2 \left(\frac{1+2s}{9} + 2 \left(\frac{1}{3} \right)^{3+s} \right) \left| \psi' \left(\frac{x_i + x_{i+1}}{2} \right) \right| + \left(\frac{s-1}{9} + \left(\frac{2}{3} \right)^{3+s} \right) |\psi'(x_{i+1})| \right). \end{aligned}$$

Hence, summing above inequality for all $i = 0, \dots, n-1$ and using property of modulus, we have the desired result. $\square$

Proposition 3.4. Let $n \in \mathbb{N}$ and $\psi : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be a differentiable function on (λ_1, λ_2) such that $\psi' \in \mathcal{L}[\lambda_1, \lambda_2]$ with $0 \leq \lambda_1 < \lambda_2$, and let $w : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be continuous and symmetric function with respect to $\frac{\lambda_1 + \lambda_2}{2}$. If $|\psi'|^q$ is s -convex in the second sense for some fixed $s \in (0, 1]$, where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} |\mathcal{R}_w(\psi, \mathcal{P})| &\leq \frac{1}{12} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(\frac{1 + 2^{1+p}}{3(1+p)} \right)^{\frac{1}{p}} \left(\frac{1}{1+s} \right)^{\frac{1}{q}} \\ &\times \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left(\left(|\psi'(x_i)|^q + \left| \psi' \left(\frac{x_i + x_{i+1}}{2} \right) \right|^q \right)^{\frac{1}{q}} + \left(\left| \psi' \left(\frac{x_i + x_{i+1}}{2} \right) \right|^q + |\psi'(x_{i+1})|^q \right)^{\frac{1}{q}} \right). \end{aligned}$$

Proof. Applying Theorem 2.6 using the same idea as in Proposition 3.3, we get the desired result. $\square$

Proposition 3.5. Let $n \in \mathbb{N}$ and $\psi : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be a differentiable function on (λ_1, λ_2) such that $\psi' \in \mathcal{L} [\lambda_1, \lambda_2]$ with $0 \leq \lambda_1 < \lambda_2$, and let $w : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$ be continuous and symmetric function with respect to $\frac{\lambda_1 + \lambda_2}{2}$. If $|\psi'|^q$ is s -convex in the second sense for some fixed $s \in (0, 1]$ and $q \geq 1$, then

$$\begin{aligned} |\mathcal{R}_w(\psi, \mathcal{P})| &\leq \frac{3}{4} \|w\|_{[\lambda_1, \lambda_2], \infty} \left(\frac{5}{54}\right)^{1-\frac{1}{q}} \left(\frac{1}{(1+s)(2+s)}\right)^{\frac{1}{q}} \\ &\times \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left(\left(\left(\frac{s-1}{9} + \left(\frac{2}{3}\right)^{3+s} \right) |\psi'(x_i)|^q + \left(\frac{1+2s}{9} + 2\left(\frac{1}{3}\right)^{3+s} \right) \left| \psi'\left(\frac{x_i + x_{i+1}}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ &\left. + \left(\left(\frac{1+2s}{9} + 2\left(\frac{1}{3}\right)^{3+s} \right) \left| \psi'\left(\frac{x_i + x_{i+1}}{2}\right) \right|^q + \left(\frac{s-1}{9} + \left(\frac{2}{3}\right)^{3+s} \right) |\psi'(x_{i+1})|^q \right)^{\frac{1}{q}} \right). \end{aligned}$$

Proof. Applying Theorem 2.10 using the same idea as in Proposition 3.3, we obtain the required result. $\square$

4. Conclusion

The main results and future research of the article can be summarized as follows:

- A new identity regarding Simpson type is established.
- New weighted Simpson type inequalities for s -convex functions in the second sense using above identity are deduced.
- Various special cases have been studied in details.
- Some applications to special means are given.
- Some results are obtained to illustrative the implementation of weighted Simpson quadrature formula.
- We hope that our results can be applied to obtain several new results in different areas of pure and applied sciences.

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