Journal of Fractional Calculus and Nonlinear Systems
jfcns.sabapub.com
ISSN : 2709-9547

J Frac Calc & Nonlinear Sys
(2024)5(2) : 33-52
doi:10.48185/jfcns.v5i2.1440

Fractional Optimal Control of Ransom Kidnapping Epidemic

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• *Received: 18 December 2024* • *Accepted: 29 December 2024* • *Published Online: 31 December 2024*

Abstract

Recently, kidnapping-for-ransom has become a primary and lucrative source of funding sustaining the activities of terror groups. Considering the menace as a social epidemic, this paper presents a fractional order mathematical model for the kidnap coinfection dynamics with recruitment and abduction occurring simultaneously. After establishing well-posedness for the positivity and boundedness, the model's two feasible equilibria (kidnap-free and persistent) are globally asymptotically stable if the kidnap propagation parameter R_k is respectively, less or bigger than unity. To minimize the menace with minimum cost of security and implementation, optimal control strategies were formulated. It is found that anti-kidnapping efforts to witch-hunt kidnappers (in their hideout) and rescue the abducted are found to be effective in addition to community vigilance.

Keywords: Kidnapping, fractional calculus, optimal control, model, Pontryaging Maximum Principle.
2010 MSC: 26A33, 34A34, 35B44, 49-XX, 93A30.

1. Introduction

Kidnapping is a predatory crime of confinement of a person against their will, perpetrated for different motives such as political, ritual, ransom, fanatical or terror ideologies. Kidnapping-for-ransom in particular has become lucrative source of funding sustaining the activities of terror groups. A social epidemic that involve severe punishment, willingness to kill the abducted, negotiations and ransoming the family of abducted, and extortion of private and social capital. Latest report [1] indicates Nigeria as the epicenter of the menace, with 2.2 million people abducted and over 2.23 trillion Naira (> \$1.2 billion) paid for ransom in one year. For nearly a decade, the country has been recording over 4,000 kidnapping cases annually, which represents 52% Africa's cases [2]. Historically [3], kidnapping in Nigeria can be trace back to early nineties when the armed group in Niger-Delta (oil producing region) started taking hostage of foreign executives of oil companies in a reaction to environmental pollution resulted by oil exploration. The country

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recorded [4, 5] first mass kidnapping of nearly 300 Chibok schoolgirls in 2014. And subsequently, the 110 Dapchi schoolgirls [6], over 300 Kankara schoolboys [7], 42 Kagara schoolboys [8], 300 Jangebe schoolgirls [9], 65 Abuja-Kaduna train passengers [10], 20 students and 2 staff of Geenfield university [11], 66 worshipers of Kaduna Baptists church [12], e.t.c.

The armed bandit groups perpetrating most of the kidnapping alongside lethal violence were basically into cattle rustling over the years [5, 13], but with the possession of sophisticated weapons (spread in Africa due to Libyan political instability) [14], they promoted to kidnapping for ransom and imposing levies on farming communities. It is estimated [15] that, there are over 100,000 weapons and ammunition in their possessions. Presently, there are over 120 armed bandit groups (demarcated by crime jurisdictions); with no fewer than 30 members in smaller and at most 2,000 members in the bigger groups. Each group has well organized leadership structure with Kachalla as general overseer taking strategic decision and coordinating other unit heads including: logistics, intelligence, informants, foot soldiers, kidnapping, cattle rustling, motorbike and camp guards [13]. The recruitment into the groups was basically on ethnic exhortation, 'yaro shiga ka kare gidanku'- calling Fulani men to join the group to protect their family [13, 16], but with the proliferation of the menace and the need to sustain the den's population, the recruitment capacity expanded to exploiting socioeconomic vulnerability coupled with poverty, unemployment and lack of education [17]. It is also said [13] that banking sector reform of reducing the need for traders to travel with cash which make road block armed robbery less lucrative, made the armed robbers changed to kidnapping for ransom.

Ransom payment (along with foodstuff, motorcycles, fuel, illicit drugs and other intoxicant) by the family of abducted make kidnapping lucrative revenue venture, with multitude beneficiaries among politicians, traditional rulers, security agents and gold miners accounted [15]. It is estimated [13] that between July 2022 and June 2023, the sum of 302 million Naira was paid in ransom out of 5 billion naira ($ 6.4 million) demanded [16], while between July 2023 and June 2024 the payment escalates to 1.1 billion Naira out of 11 billion Naira demanded. However, kidnappers killed the abducted person for his family's failure to meet their demand within the given time frame. This prompts community members volunteer for vigilance, thereby carrying out extrajudicial killing on kidnappers and their informants. It is estimated [1] that 614,937 Nigerians killed in one year.

Terrorist's ideologies spread in the same pattern as infectious diseases. In view of this, mathematical models such as [18, 19, 20, 21, 22, 23] were developed to gain an insight into the spread of ideologies. In the context of recurrent kidnapping epidemic, [24, 25] constructed mathematical that account for kidnapping as well as the recruitment of susceptible. It is suggested that hard drug dealers should first be treated as the enemies of the society, because the crime carried out under the influence of hard drugs. To curb the recruitment of young adult into banditry, [26] developed mathematical model for banditry, and then optimized media campaign as an intervention strategy. [27] categorized kidnapper's population and assessed the role of informants in kidnap propagation. [28]

modeled the menace in the sense of two strains epidemic with kidnapping propagation as one strain and adoption of abducted as other strain. It's found that atmost one strain invade the population if one the kidnap propagation numbers $\mathcal{C}_{1,2}$ is less than unity, while the two strains coexist at endemic state when both are greater than unity.

In addition to media campaign against armed bandits and the fight for illicit drugs as the basic control strategies, there is also need to avert ransom payment, supply of weapons and their possessions by unauthorized individual. These can be achieve through anti-kidnapping efforts to witch-hun kidnappers (in their hideouts) and rescue the abducted individuals. By the way, fractional order differential equation has been used to model real world phenomena with longer traits such as Covid-19 [29], Tuberclosis (TB) [30], Avian influenza [31], e.t.c., due to nonlocal property of fractional derivative operator that has the advantage of incorporating memory effect. In view of these, this paper proposes fractional order mathematical model describing the kidnap coinfection dynamics with community vigilance. Thereafter, optimal control intervention strategies to implement the aforementioned measures were formulated. However, in the sequel chapter two gives preliminaries on fractional calculus, and then followed by model formulation and the analyses in chapter three. Chapter four is the formulation of optimal control and its characterization. While Chapter five is the numerical simulation, conclusion and recommendation.

2. Preliminaries

Definition 2.1. (Left Rieman-Liouville Fractional Integral) [32]: The left R-L fractional integral I^{α}_{a+} of order $\alpha > 0$, and lower limit a of a function $x \in L^1(I, \mathbb{R})$ is defined by

$$I^{\alpha}_{a+}[x](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) d\tau \tag{2.1}$$

Definition 2.2. (Right Rieman-Liouville Fractional Integral) [32]: The right R-L fractional integral I^{α}_{b-} of order $\alpha > 0$, and upper limit b of a function $x \in L^1(I, \mathbb{R})$ is defined by

$$I^{\alpha}_{b-}[x](t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} x(\tau) d\tau \tag{2.2}$$

Definition 2.3. (Left Rieman-Liouville Fractional Derivative) [32]: The left R-L fractional derivative $D^{\alpha}_{a+}[x]$ of order $0 \leqslant \alpha \leqslant 1$ and lower limit a of a function $x \in L^1(I, \mathbb{R})$ with $I^{1-\alpha}_{a+}[x](t) \in L^1(I, \mathbb{R})$ is defined by

$$D^{\alpha}_{a+}[x](t) = \frac{d}{dt}[I^{1-\alpha}_{a+}[x]](t) \tag{2.3}$$

Definition 2.4. (Right Rieman-Liouville Fractional Derivative) [32]: The right R-L fractional derivative $D^{\alpha}_{b-}[x]$ of order $0 \leqslant \alpha \leqslant 1$ and upper limit b of a function $x \in L^1(I, \mathbb{R})$ with $I^{1-\alpha}_{b-}[x](t) \in L^1(I, \mathbb{R})$ is defined by

$$D^{\alpha}_{b-}[x](t) = -\frac{d}{dt}[I^{1-\alpha}_{b-}[x]](t) \tag{2.4}$$

Definition 2.5. (Left Caputo Fractional Derivative) [33]: The left Caputo fractional derivative ${}^{C}D^{\alpha}_{a+}[x]$ of order $0 \leqslant \alpha \leqslant 1$ and lower limit a of a function $x \in L^1(I, \mathbb{R})$ is defined by

$$ {}^{C}D^{\alpha}_{a+}[x](t) = [I^{1-\alpha}_{a+}[x']](t) \tag{2.5}$$

Definition 2.6. (Right Caputo Fractional Derivative) [33]: The right Caputo fractional derivative ${}^{C}D^{\alpha}_{b-}[x]$ of order $0 \leqslant \alpha \leqslant 1$ and upper limit b of a function $x \in L^1(I, \mathbb{R})$ is defined by

$$ {}^{C}D^{\alpha}_{b-}[x](t) = -[I^{1-\alpha}_{b-}[x']](t) \tag{2.6}$$

Remark 2.7. For $[x - x(a)] \in \mathcal{AC}^{\alpha}_{a+}(I, \mathbb{R})$ and $[x - x(b)] \in \mathcal{AC}^{\alpha}_{b-}(I, \mathbb{R})$, the following relation hold

$$ {}^{C}D^{\alpha}_{a+}[x](t) = D^{\alpha}_{a+}[x - x(a)](t) \tag{2.7}$$

$$ {}^{C}D^{\alpha}_{b-}[x](t) = D^{\alpha}_{b-}[x - x(b)](t) \tag{2.8}$$

Definition 2.8. (Fractional Optimal Control) [34]: Let $a < b$ be two real numbers. Let $m, n, j \in \mathbb{N}^*$, let $0 < \alpha \leqslant 1$ and $\beta \geqslant \alpha$ be fixed. The Caputo optimal control problem of Bolza form is given by

$$\begin{cases} \text{minimize} & \Phi(x(a),\ x(b)) + I^{\beta}_{a+}F[(x, u, .)] \\ \text{subject to} & x \in_c \mathbb{AC}^{\alpha}_{a+}([a, b], \mathbb{R}^n),\ u \in L^{\infty}([a, b], \mathbb{R}^m), \\ & {}^{C}D^{\alpha}_{a+}[x](t) = f(x(t), u(t), t), \\ & g(x(a),\ x(b)) \in C \\ & u(t) \in U,\ a.e.t \in [a, b]. \end{cases} \tag{2.9}$$

Theorem 2.9. *(Generalized Mean Value Theorem) [35]: Suppose that* $x(t) \in C[a,\ b]$ *and* ${}^{C}D^{\alpha}x(t) \in C(a,\ b]$, *for* $0 < \alpha \leqslant 1$, *then*

$$x(t) = x(a) + \frac{1}{\Gamma(\alpha)}\ {}^{C}D^{\alpha}x(\xi)\ (t - a)^{\alpha}, \qquad a < \xi < t,\ \ \forall\, t \in (a,\ b] \tag{2.10}$$

Remark 2.10. If $x(t) \in C[0,\ b]$ and ${}^{C}D^{\alpha}x(t) \in C(0,\ b]$, for $0 < \alpha \leqslant 1$. It is clear from Theorem 2.9 that if ${}^{C}D^{\alpha}x(t) \geqslant 0$, $\forall\, t \in (0,\ b]$, then the function $x(t)$ is non-decreasing and if ${}^{C}D^{\alpha}x(t) \leqslant 0$, $\forall\, t \in (0,\ b]$ then the function $x(t)$ is non-increasing for all $t \in [0,\ b]$.

Lemma 2.11. *(Laplace Transform of Caputo Derivative Operator): The Laplace transform of Caputo fractional derivative operator for* $0 < \alpha < 1$ *is derived as*

$$L\{D^{\alpha}x(t)\} = s^{\alpha}L\{x(t)\} - s^{\alpha-1}x(0). \tag{2.11}$$

Theorem 2.12. *(Local Stability) [36]: The equilibrium solution* x^0 *of Caputo fractional system*

$$ {}^{C}D^{\alpha}_{0}[x](t) = f(t, x) \tag{2.12}$$

is locally asymptotically stable if all the eigenvalues $\lambda_j, j = 1, 2, \ldots, n$ *of the Jacobian matrix* $\left[\frac{\partial f_i}{\partial x_j}\right]$, $i, j = 1, 2, \ldots, n$ *evaluated at equilibrium* x^0 *satisfy*

$$|\arg(\lambda_j)| > \frac{\alpha\pi}{2}, \forall\, j, \text{ and } \alpha \in (0,1). \tag{2.13}$$

Theorem 2.13. *(Global Stability) [37]: Let* $x^0 = 0$ *be an equilibrium for system (2.12) and* $\Omega \subseteq \mathbb{R}^n$ *be a domain containing* $x^0 = 0$. *Let*

$$V(t,x) : [0,\infty) \times \Omega \longrightarrow \mathbb{R} \tag{2.14}$$

be a continuously differentiable function such that

$$W_1(x) \leqslant V(t,x) \leqslant W_2(x), \tag{2.15}$$

$${}^{C}D^{\alpha}V(t,x) \leqslant -W_3(x), \quad \forall\, t \geqslant 0, \quad \forall\, x \in \Omega, \quad 0 < \alpha < 1. \tag{2.16}$$

where $W_1(x), W_2(x)$ *and* $W_3(x)$ *are continuous positive definite functions on* Ω, *then* $x_e = 0$ *is globally asymptotically stable.*

Lemma 2.14. *[38] Let* $x(t) \in \mathbb{R}$ *be continuous and derivable function, then for any time instant* $t \geqslant t_0$

$$\frac{1}{2}\, {}^{C}D^{\alpha}x^2(t) \leqslant x(t)\ {}^{C}D^{\alpha}x(t), \qquad \forall \alpha \in (0,1). \tag{2.17}$$

Theorem 2.15. *(Pontryaging Maximum Principle) [34]: Assume* $(x^*, u^*) \in_c \mathbb{AC}^{\alpha}_{a+}([a,b], \mathbb{R}^n) \times L^{\infty}([a,b], \mathbb{R}^m)$ *is an optimal solution to problem (2.9). Then there exists a non-trivial couple* (p, p^0), *where* $p \in_c \mathbb{AC}^{\alpha}_{a+}([a,b], \mathbb{R}^n)$ *(called an adjoint vector) and* $p^0 \leqslant 0$, *such that the following conditions hold:*

1. *Fractional Hamiltonian system (extremal equations):*

$${}^{C}D^{\alpha}_{a+}[x^*](t) = \partial H_3(x^*(t), u^*(t), p(t), p^0, t) \tag{2.18}$$

$$D^{\alpha}_{b-}[p](t) = \partial H_1(x^*(t), u^*(t), p(t), p^0, t) \tag{2.19}$$

   for almost every $t \in [a,b]$, *where the Hamiltonian* $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times [a,b] \longrightarrow \mathbb{R}$ *associated to (2.9) is defined by*

$$H(x,u,p,p^0,t) = \langle p, f(x,u)\rangle_{\mathbb{R}^n} + p^0 \frac{(b-t)^{\beta-1}}{\Gamma(\beta)} F(x,u,t) \tag{2.20}$$

   for all $(x,u,p,p^0,t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times [a,b]$*; and*

2. *Hamiltonian maximization condition:*

$$u^*(t) \in \operatorname{argmax}_v H(x^*(t), u^*(t), p(t), p^0, t) \qquad \text{for} \quad \text{a.e.} \quad t \in [a,b];$$

3. *Transversality condition on the adjoint vector: if in addition* g *is submersive at* $(x^*(a), x^*(b))$, *then the nontrivial couple* p, p^0 *can be selected to satisfy*

$$I_{b-}^{1-\alpha}[p](a) = -p^0\ \partial_1\Phi(x^*(a), x^*(b)) - \partial_1\ g(x^*(a), x^*(b))^T \times \Psi \tag{2.21}$$

$$I_{b-}^{1-\alpha}[p](b) = -p^0\ \partial_2\Phi(x^*(a), x^*(b)) + \partial_2\ g(x^*(a), x^*(b))^T \times \Psi \tag{2.22}$$

where - $\Psi \in N_C[g(x^*(a), x^*(b))]$.

Remark 2.16. Assume that the Hamiltonian consider in theorem 2.15 is differentiable with respect to its second variable (for example, if f and F are so). And,

1. if U is convex, then the Hamiltonian maximization condition in theorem 2.15 implies the (weaker) nonnegative Hamiltonian gradient condition given by

$$\langle\ \partial H_2(x^*(t), u^*(t), p(t), p^0, t),\ v - u^*(t)\ \rangle_{\mathbb{R}^m}\ \leqslant 0, \quad \forall\, v \in U \text{and a.e. } t \in [a, b]. \tag{2.23}$$

2. similarly, if $U = \mathbb{R}^m$ (that is, no control constraint in problem (2.9)), then the Hamiltonian maximization condition in theorem 2.15 implies the (weaker) null Hamiltonian gradient condition given by

$$\partial H_2(x^*(t), u^*(t), p(t), p^0, t) = 0_{\mathbb{R}^m} \qquad \text{for a.e. } t \in [a, b] \tag{2.24}$$

3. Model Formulation

Considering kidnapper's activities in the sense of coinfection with recruitment and abduction occurring simultaneously as described by figure 1, the total human population N is subdivided into vulnerable to recruitment V, susceptible to abduction S, kidnappers K, individuals in the correctional center C and in the abducted confinement A.

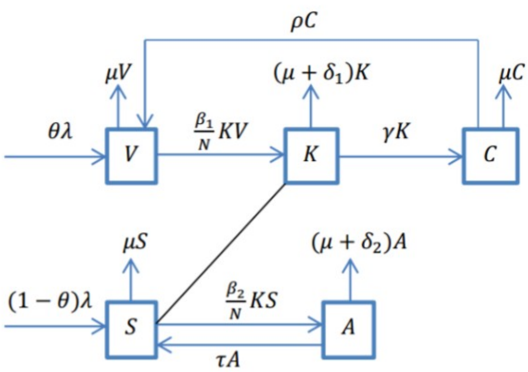

Figure 1: Schematic diagram of kidnap coinfection

It is assumed that new born of is human either becomes vulnerable to recruitment or susceptible to kidnapping at rates $\theta\lambda$ and $(1-\theta)\lambda$ respectively. The vulnerable population also increases with the relapse ρ of individuals released from correctional center and decreases at natural mortality rate μ. It also decreases and converts into kidnappers subpopulation following the recruitment at rate β_1. Moreover, kidnappers subpopulation decreases at the kidnap induced death rate δ_1, natural death rate μ or progression to correctional center at rate γ. However, the susceptible population increases with abducted rescued at rate τ and decreases due to natural death μ and kidnappers induced death rate β_2. Furthermore, the number of individuals in the abducted camp decreases due to natural death rate μ and kidnappers induced death rate δ_2. Therefore, the kidnap coinfection dynamic is described by nonlinear system of fractional order differential equations in the Caputo sense:

$$ {}^{C}D^{\alpha}V = \theta\lambda^{\alpha} + \rho^{\alpha}C - \frac{\beta_1^{\alpha}}{N}KV - \mu^{\alpha}V, \tag{3.1} $$

$$ {}^{C}D^{\alpha}K = \frac{\beta_1^{\alpha}}{N}KV - (\mu^{\alpha} + \delta_1^{\alpha} + \gamma^{\alpha})K, \tag{3.2} $$

$$ {}^{C}D^{\alpha}C = \gamma^{\alpha}K - (\mu^{\alpha} + \rho^{\alpha})C, \tag{3.3} $$

$$ {}^{C}D^{\alpha}S = (1-\theta)\lambda^{\alpha} + \tau^{\alpha}A - \frac{\beta_2^{\alpha}}{N}KS - \mu^{\alpha}S, \tag{3.4} $$

$$ {}^{C}D^{\alpha}A = \frac{\beta_2^{\alpha}}{N}KS - (\mu^{\alpha} + \delta_2^{\alpha} + \tau^{\alpha})A. \tag{3.5} $$

Remark 3.1. According to [39], the fractional derivative operator ${}^{C}D^{\alpha}$ has a dimension $(\text{time})^{-\alpha}$ instead of $(\text{time})^{-1}$, so that due to dimensional analysis, on the right hand side of (3.1) - (3.5) the unit parameters $\lambda, \beta_1, \beta_2, \mu, \rho, \delta_1, \delta_2, \gamma, \tau$ must have power α.

Responding to situation by volunteer vigilante made kidnappers more sensitive while reducing their mingling (contacts) prompting to recruitment and abduction. Different functions have been used to incorporate reduction of contact with infection in epidemic models such as: e^{-mI}, $\frac{1}{1+mI}$, $\frac{1}{1+mI^2}$, $1 - \frac{1}{m+I}$ [40] and $c(I) = c_1 - c_2 f(I)$ where $f(0) = 0$, $f'(I) = 0$, $\lim_{t\to\infty} f(I) = 1$ [41, 42] because they both decrease rapidly as I increases. Here, since the rates of recruitment β_1 and abduction β_2 depend on contact rate c, and the respective probabilities p_1 and p_2. Assume random mixing between individuals such that $c = rN$ and define a contact factor to be a function of the number of kidnappers $r(K) = \frac{m}{1+aK^2}$, where m is the maximum reduced contact rate due to vigilance, and $a \geqslant 0$ measures the efficacy of the vigilance. By substituting this rate for $\beta_1 = p_1 c$ and $\beta_2 = p_2 c$, the system (3.1) - (3.5) becomes

$$ {}^{C}D^{\alpha}V = \theta\lambda^{\alpha} + \rho^{\alpha}C - \frac{b_1^{\alpha}K}{1+aK^2}V - \mu^{\alpha}V, \tag{3.6} $$

$$ {}^{C}D^{\alpha}K = \frac{b_1^{\alpha}K}{1+aK^2}V - (\mu^{\alpha} + \delta_1^{\alpha} + \gamma^{\alpha})K, \tag{3.7} $$

$$ {}^{C}D^{\alpha}C = \gamma^{\alpha}K - (\mu^{\alpha} + \rho^{\alpha})C, \tag{3.8} $$

$$ {}^{C}D^{\alpha}S = (1-\theta)\lambda^{\alpha} + \tau^{\alpha}A - \frac{b_2^{\alpha}K}{1+aK^2}S - \mu^{\alpha}S, \tag{3.9} $$

$$ {}^{C}D^{\alpha}A = \frac{b_2^{\alpha}K}{1+aK^2}S - (\mu^{\alpha} + \delta_2^{\alpha} + \tau^{\alpha})A, \tag{3.10} $$

such that

$$V(0),\ K(0),\ C(0),\ S(0),\ A(0),\ \geqslant 0,$$

where for convenient $b_1 = mp_1$ and $b_2 = mp_2$.

3.1. Well-Posedness

To show the criminological feasibilty of the model, it suffices to establish positivity and boundedness.

Theorem 3.2. *The set* $\mathbb{R}^5_+$ *is positively invariant w.r.t. system (3.6) - (3.10). Furthermore, all the solutions are confined in a bounded subset* $\Omega = \left\{(V, K, C, S, A) \in \mathbb{R}^5_+ : 0 \leqslant V, K, C, S, A \leqslant \frac{\lambda^\alpha}{\mu^\alpha}\right\}$

Proof

Let $(V(0), K(0), C(0), S(0), A(0)) \in C[0, h]$ and $({}^C D^\alpha V(0), {}^C D^\alpha K(0), {}^C D^\alpha C(0), {}^C D^\alpha S(0), {}^C D^\alpha A(0)) \in C(0, h]$ then by theorem 2.9

$$V(t) = V(0) + \frac{1}{\Gamma(\alpha)}\ {}^C D^\alpha V(\tau)\ (t)^\alpha, \tag{3.11}$$

$$K(t) = K(0) + \frac{1}{\Gamma(\alpha)}\ {}^C D^\alpha K(\tau)\ (t)^\alpha, \tag{3.12}$$

$$C(t) = C(0) + \frac{1}{\Gamma(\alpha)}\ {}^C D^\alpha C(\tau)\ (t)^\alpha, \tag{3.13}$$

$$S(t) = S(0) + \frac{1}{\Gamma(\alpha)}\ {}^C D^\alpha S(\tau)\ (t)^\alpha, \tag{3.14}$$

$$A(t) = A(0) + \frac{1}{\Gamma(\alpha)}\ {}^C D^\alpha A(\tau)\ (t)^\alpha, \qquad \tau \in [0, h]. \tag{3.15}$$

Also, the system (3.6) - (3.10) yields

$${}^C D^\alpha V(t)|_{V=0} = \theta\lambda^\alpha + \rho^\alpha C(t) \geqslant 0, \tag{3.16}$$

$${}^C D^\alpha K(t)|_{K=0} = 0, \tag{3.17}$$

$${}^C D^\alpha C(t)|_{C=0} = \gamma^\alpha K \geqslant 0, \tag{3.18}$$

$${}^C D^\alpha S(t)|_{S=0} = (1-\theta)\lambda^\alpha + \tau^\alpha A \geqslant 0, \tag{3.19}$$

$${}^C D^\alpha A(t)|_{A=0} = \frac{b_2^\alpha K}{1 + aK^2} S \geqslant 0. \tag{3.20}$$

The remark 2.10 and (3.16) - (3.20) imply that (3.11) - (3.15) are non-decreasing. Hence, (V, K, C, S, A) cannot escape from hyperplane V=K=C=S=A=0. Also, on each hyperplane bounding the non-negative orthant, therefore the domain $\mathbb{R}^5_+$, i.e., is positively invariant.

For the boundedness part, apply Caputo operator to

$$N = V + K + C + S + A$$

and substitute (3.6) - (3.10) to get

$$\begin{aligned} {}^{C}D^{\alpha}N &= \lambda^{\alpha} - \mu^{\alpha}N - \delta_1^{\alpha}K - \delta_2^{\alpha}A \\ &\leqslant \lambda^{\alpha} - \mu^{\alpha}N. \end{aligned} \tag{3.21}$$

by applying Laplace transform to (3.21) and using (2.11) gives

$$\mathcal{L}\{N(t)\} \leqslant \frac{\lambda^{\alpha}}{s(s^{\alpha}+\mu^{\alpha})} + \frac{s^{\alpha-1}}{s^{\alpha}+\mu^{\alpha}}N(0). \tag{3.22}$$

Subsequent to partial fraction and Taylor series decomposition of $(1+\frac{\mu^{\alpha}}{s^{\alpha}})^{-1}$, (3.22) becomes

$$\mathcal{L}\{N(t)\} \leqslant \frac{\lambda^{\alpha}}{\mu^{\alpha}s} + \left(N(0) - \frac{\lambda^{\alpha}}{\mu^{\alpha}}\right)\sum_{m=0}^{\infty}\frac{(-\mu^{\alpha})^m}{s^{\alpha m+1}}. \tag{3.23}$$

Taking the inverse Laplace transform of (3.23) yields solution in terms of Mittag-Leffler function (series)

$$N(t) \leqslant \frac{\lambda^{\alpha}}{\mu^{\alpha}} + \left(N(0) - \frac{\lambda^{\alpha}}{\mu^{\alpha}}\right)\sum_{m=0}^{\infty}\frac{(-\mu t)^{\alpha m}}{\Gamma(\alpha m+1)}. \tag{3.24}$$

By asymptotic decay (refer to [32]) of Mittag-Leffler (the generalized exponential function) , the long term behaviour gives

$$\limsup_{t\to\infty} N(t) \leqslant \frac{\lambda^{\alpha}}{\mu^{\alpha}}.$$

Hence the criminological feasibility of the model.

3.2. Equilibria and Stability

By equating (3.6) - (3.10) to zero in the absence of Kidnapper $(K=0)$, the kidnap free equilibrium KFE is obtained as

$$\begin{aligned} E^0 &= (V^0, K^0, C^0, S^0, A^0) \\ &= \left(\frac{\theta\lambda^{\alpha}}{\mu^{\alpha}},\ 0,\ 0,\ \frac{(1-\theta)\lambda^{\alpha}}{\mu^{\alpha}},\ 0\right) \end{aligned}$$

It is consistent to reality that in the absence of kidnappers, no one confined in the abducted camp. Moreover, to find kidnap propagation parameter R_k using the method of next generation matrix [44] rewrite the kidnappers state as a sum of recruitment and transition

$${}^{C}D^{\alpha}K = \mathcal{F} + \mathcal{T}, \tag{3.25}$$

where

$$\mathcal{F} = \frac{b_1^{\alpha}K}{1+aK^2}V, \qquad \mathcal{T} = -(\mu^{\alpha} + \delta_1^{\alpha} + \gamma^{\alpha})K.$$

Evaluating their partial derivatives (w.r.t. K) at E^0 gives

$$F_0 = \frac{\theta\lambda^\alpha b_1^\alpha}{\mu^\alpha}, \qquad T_0 = -\mu^\alpha - \delta_1^\alpha - \gamma^\alpha.$$

As in [43], the kidnap propation parameter R_k is define as the spectral radius $\rho(-F_0 T_0^{-1})$. So,

$$R_k = \frac{\theta\lambda^\alpha b_1^\alpha}{\mu^\alpha(\mu^\alpha + \delta_1^\alpha + \gamma^\alpha)}. \tag{3.26}$$

However, equating (3.6) - (3.10) to zero in the presence of kidnapper $(K \neq 0)$, the kidnap persistent equilibrium $E^* = (V^*, K^*, C^*, S^*, A^*)$ is obtained as

$$\begin{aligned}
V^* &= \frac{\theta\lambda^\alpha}{\mu^\alpha R_k}[1 + a(K^*)^2],\\
C^* &= \frac{\gamma^\alpha}{\mu^\alpha + \rho^\alpha}K^*,\\
S^* &= \frac{(1-\theta)\lambda^\alpha(\mu^\alpha + \delta_2^\alpha + \tau^\alpha)[1 + a(K^*)^2]}{b_2(\mu^\alpha + \delta_2^\alpha)K^* + \mu^\alpha(\mu^\alpha + \delta_2^\alpha + \tau^\alpha)[1 + a(K^*)^2]},\\
A^* &= \frac{b_2(1-\theta)\lambda^\alpha K^*}{b_2(\mu^\alpha + \delta_2^\alpha)K^* + \mu^\alpha(\mu^\alpha + \delta_2^\alpha + \tau^\alpha)[1 + a(K^*)^2]},
\end{aligned}$$

where

$$K_{1,2}^* = \frac{-\mathcal{B} \pm \sqrt{\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}}}{2\mathcal{A}} \tag{3.27}$$

with

$$\begin{aligned}
\mathcal{A} &= a\theta\lambda^\alpha(\mu^\alpha + \rho^\alpha),\\
\mathcal{B} &= R_k[\rho^\alpha(\mu^\alpha + \delta_1^\alpha) + \mu^\alpha(\mu^\alpha + \delta_1^\alpha + \gamma^\alpha)],\\
\mathcal{C} &= -\theta\lambda^\alpha(R_k - 1)(\mu^\alpha + \rho^\alpha).
\end{aligned}$$

Since for $R_k > 1$, the descriminant

$$\begin{aligned}
\Delta &= B^2 - 4AC\\
&= R_k^2[\rho^\alpha(\mu^\alpha + \delta_1^\alpha) + \mu^\alpha(\mu^\alpha + \delta_1^\alpha + \gamma^\alpha)]^2 + 4\theta^2(\lambda^\alpha)^2 a(R_k - 1)(\mu^\alpha + \rho^\alpha)^2\\
&> 0
\end{aligned}$$

then

$$K^* = \frac{-\mathcal{B} + \sqrt{\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}}}{2\mathcal{A}} > 0$$

is the only feasible state. The following theorems analyze the stability of the equilibria.

Theorem 3.3. *The kidnap free equilibrium is globally asymptotically stable if* $R_k < 1$.

Proof

In analogy with quadratic-type Lyapunov function [45], define

$$L : \left\{ (V, K, C, S, A) \in \Omega : 0 \leqslant V, K, C, S, A \leqslant \frac{\lambda^\alpha}{\mu^\alpha} \right\} \longrightarrow \mathbb{R}$$

by

$$L(V, K, C, S, A) = \frac{1}{2}(V - V^0 + K + C + S - S^0 + A)^2. \tag{3.28}$$

Clearly, L is well-define positive definite i.e

$$L(V^0, K^0, C^0, S^0, A^0) = 0,$$
$$L(V, K, C, S, A) > 0 \qquad elsewhere.$$

Moreover,

$${}^C D^\alpha L(V, K, C, S, A) = \frac{1}{2}\, {}^C D^\alpha (V - V^0 + K + C + S - S^0 + A)^2. \tag{3.29}$$

By lemma 2.14

$${}^C D^\alpha L \leqslant (V - V^0 + K + C + S - S^0 + A)\, {}^C D^\alpha (V - V^0 + K + C + S - S^0 + A).$$

By the linearity of Caputo operator and using (3.6) - (3.10) gives

$${}^C D^\alpha L \leqslant (N - N^0)(\lambda^\alpha - \mu^\alpha N - \delta_1^\alpha K - \delta_2^\alpha A). \tag{3.30}$$

At E^0, $\quad \lambda^\alpha = \mu^\alpha N^0$

$${}^C D^\alpha L \leqslant -(N - N^0)[\mu^\alpha (N - N^0) + \delta_1^\alpha K + \delta_2^\alpha A].$$

Also the boundedness established that $N \leqslant \frac{\lambda^\alpha}{\mu^\alpha} = N^0$, therefore $N - N^0 \leqslant 0$. So

$${}^C D^\alpha L \leqslant 0$$

Hence theorem 2.13 is satisfied. On the other hand, evaluating the Jacobian Matrix of system (3.6) - (3.10) at E^0 gives

$$J^0 = \begin{pmatrix} -\mu^\alpha & -\frac{b_1\theta\lambda^\alpha}{\mu^\alpha} & \rho^\alpha & 0 & 0 \\ 0 & \frac{b_1\theta\lambda^\alpha}{\mu^\alpha} - \mu^\alpha - \delta_1^\alpha - \gamma^\alpha & 0 & 0 & 0 \\ 0 & \gamma^\alpha & -\mu^\alpha - \rho^\alpha & 0 & 0 \\ 0 & -\frac{b_2(1-\theta)\lambda^\alpha}{\mu^\alpha} & 0 & -\mu^\alpha & \tau^\alpha \\ 0 & \frac{b_2(1-\theta)\lambda^\alpha}{\mu^\alpha} & 0 & 0 & -\mu^\alpha - \delta_2^\alpha - \tau^\alpha \end{pmatrix}$$

whose characteristics equation $|J^0 - \Lambda I| = 0$ yields eigenvalues :
$\Lambda_1 = -\mu^\alpha$, $\Lambda_2 = \frac{b_1\theta\lambda^\alpha}{\mu^\alpha} - \mu^\alpha - \delta_1^\alpha - \gamma^\alpha$, $\Lambda_3 = -\mu^\alpha - \rho^\alpha$, $\Lambda_4 = -\mu^\alpha$, $\Lambda_5 = -\mu^\alpha - \delta_2^\alpha - \tau^\alpha$,.
Clearly, all the eigenvalues have negative real part with the exception of Λ_2. For E^0 to be stable, satisfying theorem 2.12 it is required that $\Lambda_2 < 0$, which implies $\frac{b_1\theta\lambda^\alpha}{\mu^\alpha(\mu^\alpha + \delta_1^\alpha + \gamma^\alpha)} < 1$ or $R_k < 1$.

Theorem 3.4. *The kidnap persistent equilibrium is globally asymptotically stable if* $R_k > 1$.

Proof

Define Lyapunov function

$$L : \left\{(V, K, C, S, A) \in \Omega : 0 \leqslant V, K, C, S, A \leqslant \frac{\lambda^\alpha}{\mu^\alpha}\right\} \longrightarrow \mathbb{R}$$

by

$$L(V, K, C, S, A) = \frac{1}{2}(V - V^* + K - K^* + C - C^* + S - S^* + A - A^*)^2 \tag{3.31}$$

clearly, L is well-define positive definite i.e

$$\begin{aligned} L(V^*, K^*, C^*, S^*, A^*) &= 0, \\ L(V, K, C, S, A) &> 0 \quad \textit{elsewhere}. \end{aligned}$$

Moreover,

$$^C D^\alpha L(V, K, C, S, A) = \frac{1}{2}\, ^C D^\alpha (V - V^* + K - K^* + C - C^* + S - S^* + A - A^*)^2 \tag{3.32}$$

By lemma 2.14

$$\begin{aligned} ^C D^\alpha L \leqslant\ & (V - V^* + K - K^* + C - C^* + S - S^* + A - A^*) \\ & \times ^C D^\alpha (V - V^* + K - K^* + C - C^* + S - S^* + A - A^*). \end{aligned}$$

By linearity of Caputo derivative operator and using (3.6) - (3.10)

$$^C D^\alpha L \leqslant (N - N^*)(\lambda^\alpha - \mu^\alpha N - \delta_1^\alpha K - \delta_2^\alpha A). \tag{3.33}$$

Since $N \leqslant \frac{\lambda^\alpha}{\mu^\alpha}$, then

$$^C D^\alpha L \leqslant -(N - N^*)\,(\delta_1^\alpha K + \delta_2^\alpha A). \tag{3.34}$$

At E^*,

$$N^* = \frac{\theta\lambda^\alpha[1 + a(K^*)^2]}{\mu^\alpha R_k} + \frac{\mu^\alpha + \rho^\alpha + \gamma^\alpha}{\mu^\alpha + \rho^\alpha} K^* + \frac{(1-\theta)\lambda^\alpha[(\mu^\alpha + \delta_2^\alpha + \tau^\alpha)[1 + a(K^*)^2] + b_2 K^*]}{b_2(\mu^\alpha + \delta_2^\alpha)K^* + \mu^\alpha(\mu^\alpha + \delta_2^\alpha + \tau^\alpha)[1 + a(K^*)^2]}$$

so,

$$\begin{aligned} N - N^* &\leqslant \frac{\lambda^\alpha}{\mu^\alpha} - \frac{\theta\lambda^\alpha[1 + a(K^*)^2]}{\mu^\alpha R_k} - \frac{\mu^\alpha + \rho^\alpha + \gamma^\alpha}{\mu^\alpha + \rho^\alpha} K^* - \frac{(1-\theta)\lambda^\alpha[(\mu^\alpha + \delta_2^\alpha + \tau^\alpha)[1 + a(K^*)^2] + b_2 K^*]}{b_2(\mu^\alpha + \delta_2^\alpha)K^* - \mu^\alpha(\mu^\alpha + \delta_2^\alpha + \tau^\alpha)[1 + a(K^*)^2]} \\ &\leqslant \frac{\lambda^\alpha}{\mu^\alpha} - \frac{\theta\lambda^\alpha}{\mu^\alpha R_k} \\ &= \frac{\lambda^\alpha}{\mu^\alpha}(R_k - \theta) \\ &> 0, \quad \text{for} \quad 0 \leqslant \theta \leqslant 1,\ R_k > 1. \end{aligned}$$

therefore,

$$^{C}D^{\alpha}L \leqslant 0. \tag{3.35}$$

Hence theorem 2.13 is satisfied. Moreover, evaluating the Jacobian Matrix of system (3.6) - (3.10) at E^* gives

$$J^* = \begin{pmatrix} -\frac{b_1K^*}{1+a(K^*)^2} - \mu^{\alpha} & -\frac{b_1\theta\lambda^{\alpha}[1-a(K^*)^2]}{\mu^{\alpha}R_k[1+a(K^*)^2]} & \rho^{\alpha} & 0 & 0 \\ \frac{b_1K}{1+a(K^*)^2} & \frac{b_1\theta\lambda^{\alpha}[1-a(K^*)^2]}{\mu^{\alpha}R_k[1+a(K^*)^2]} - \mu^{\alpha} - \delta_1^{\alpha} - \gamma^{\alpha} & 0 & 0 & 0 \\ 0 & \gamma^{\alpha} & -\mu^{\alpha} - \rho^{\alpha} & 0 & 0 \\ 0 & -p & 0 & -\frac{b_2K^*}{1+a(K^*)^2} - \mu^{\alpha} & \tau^{\alpha} \\ 0 & p & 0 & \frac{b_2K^*}{1+a(K^*)^2} & -\mu^{\alpha} - \delta_2^{\alpha} - \tau^{\alpha} \end{pmatrix}$$

where
$p = \frac{(1-\theta)\lambda^{\alpha}b_2(\mu^{\alpha}+\delta_2^{\alpha}+\tau^{\alpha})[1-a(K^*)^2]}{[1+a(K^*)^2]\{b_2(\mu^{\alpha}+\delta_2^{\alpha})K^*+\mu^{\alpha}(\mu^{\alpha}+\delta_2^{\alpha}+\tau^{\alpha})[1+a(K^*)^2]\}}$.
So, the characteristics equation $|J^* - \Lambda I| = 0$ yields

$$\Lambda_{1,2} = \frac{-b_2K^* - (2\mu^{\alpha} + \delta_2^{\alpha} + \tau^{\alpha})[1 + a(K^*)^2] \pm \sqrt{\Delta}}{2[1 + a(K^*)^2]}$$

and

$$q_3\Lambda^3 + q_2\Lambda^2 + q_1\Lambda + q_0 = 0 \tag{3.36}$$

where
$\Delta = [(\delta_2^{\alpha} + \tau^{\alpha})(1 + a_2K^2) - b_2K]^2 + 4b_2K^*[1 + a(K^*)^2] > 0,$

$$\begin{aligned} q_3 &= 1, \\ q_2 &= \frac{\mu^{\alpha}R_0[b_1K^* + \mu^{\alpha}(1 + a(K^*)^2)] + 2b_1\theta\lambda^{\alpha}a(K^*)^2 + \mu^{\alpha}R_0(\mu^{\alpha} + \rho^{\alpha})[1 + a(K^*)^2]}{\mu^{\alpha}R_0[1 + a(K^*)^2]}, \\ q_1 &= \frac{\mu^{\alpha}R_0(\mu^{\alpha} + \rho^{\alpha})\{b_1K^* + \mu^{\alpha}[1 + a(K^*)^2]\} + b_1\theta\lambda^{\alpha}K^*[2a(\mu^{\alpha} + \rho^{\alpha})K^* + b_1]}{\mu^{\alpha}R_0[1 + a(K^*)^2]}, \\ q_0 &= \frac{b_1[\mu^{\alpha}(\mu^{\alpha} + \delta_1^{\alpha} + \gamma^{\alpha}) + \rho^{\alpha}(\mu^{\alpha} + \delta_1^{\alpha})]K^* + 3a\mu^{\alpha}(\mu^{\alpha} + \delta_1^{\alpha} + \gamma^{\alpha})(\mu^{\alpha} + \delta_1^{\alpha})(K*)^2}{1 + a(K^*)^2}. \end{aligned}$$

Since

$$\begin{aligned} q_1q_2 - q_0q_3 = q_1 &\frac{\mu^{\alpha}R_0[b_1K^* + \mu^{\alpha}(1 + a(K^*)^2)] + 2b_1\theta\lambda^{\alpha}a(K^*)^2}{\mu^{\alpha}R_0[1 + a(K^*)^2]} \\ &+ \frac{(\mu^{\alpha} + \rho^{\alpha})^2[b_1K^* + \mu^{\alpha}[1 + a(K^*)^2] + 2a(\mu^{\alpha} + \delta_1^{\alpha} + \gamma^{\alpha})(K^*)^2] + b_1\rho^{\alpha}\gamma^{\alpha}K^*}{1 + a(K^*)^2} > 0 \end{aligned}$$

meanwhile there is no sign change between the entries $q_3, q_2, \frac{q_1q_2-q_0q_3}{q_2}, q_0$ forming the first column of Routh array [47], then by Routh-Hurtwitz stability criterion [46, 47], all the roots of the polynomial (3.36) lie in the left half plane. Moreover, all the eigenvalues $\Lambda_i, \quad i = 1, \ldots, 5$ have negetive real part with the exception of

$$\Lambda_2 = \frac{-b_2K^* - (2\mu^{\alpha} + \delta_2^{\alpha} + \tau^{\alpha})[1 + a(K^*)^2] + \sqrt{\Delta}}{2[1 + a(K^*)^2]}$$

Therefore, for E^* to be stable satisfying theorem 2.12, it is required that $\Lambda_2 < 0$, which implies

$$\begin{aligned} \sqrt{\Delta} &< b_2 K^* + (2\mu^\alpha + \delta_2^\alpha + \tau^\alpha)[1 + a(K^*)^2] \\ \iff \quad -4\{b_2(\mu^\alpha + \delta_2^\alpha)K^* &+ \mu^\alpha(\mu^\alpha + \delta_2^\alpha + \tau^\alpha)[1 + a(K^*)^2]\} < 0 \\ \iff \quad b_2(\mu^\alpha + \delta_2^\alpha)K^* &+ \mu^\alpha(\mu^\alpha + \delta_2^\alpha + \tau^\alpha)[1 + a(K^*)^2] > 0 \end{aligned} \tag{3.37}$$

$\iff \quad K^* > 0$ or $R_k > 1$.

4. Formulation of Optimal Control

The successful intervention strategy is one which decrease the number of infections while minimizing the associated cost [42, 48]. Kidnappers use phone call to bargain ransom with abducted's family. Meanwhile, the security tracking system can be used to find their location and witch-hunt them in their hideout thereby rescuing the abducted to avert ransom payment that proliferate their activities. To incorporate these with a minimum cost of security and implementation, the time dependent control variables $u_1(t), u_2(t)$ are incorporated to the model (3.6) - (3.10) as

$$^{C}D^\alpha V = \theta\lambda^\alpha + \rho^\alpha C - \frac{b_1^\alpha K}{1 + aK^2} V - \mu^\alpha V, \tag{4.1}$$

$$^{C}D^\alpha K = \frac{b_1^\alpha K}{1 + aK^2} V - [\mu^\alpha + \delta_1^\alpha + \gamma^\alpha(1 + u_1)]K, \tag{4.2}$$

$$^{C}D^\alpha C = \gamma^\alpha(1 + u_1)K - (\mu^\alpha + \rho^\alpha)C, \tag{4.3}$$

$$^{C}D^\alpha S = (1 - \theta)\lambda^\alpha + \tau^\alpha(1 + u_2)A - \frac{b_2^\alpha K}{1 + aK^2} S - \mu^\alpha S, \tag{4.4}$$

$$^{C}D^\alpha A = \frac{b_2^\alpha K}{1 + aK^2} S - [\mu^\alpha + \delta_2^\alpha + \tau^\alpha(1 + u_2)]A. \tag{4.5}$$

where

- $u_1(t) \in [0, u_{1max}]$ represents anti-kidnapping effort to witch-hunt kidnappers in their hideout,
- $u_2(t) \in [0, u_{2max}]$ represents anti-kidnapping effort to rescue abducted individuals to avert ransom payment,

with the set of admissible control functions defined by

$$U = \{u_i(t) : u_i(t) \in [0, 1] \text{ Lebesgue measurable}, i = 1, 2, \ 0 \leqslant t \leqslant t_f\}.$$

Note that in the absence of controls $u_1 = u_2 = 0$, the control system (4.1) - (4.5) reduces to kidnap coinfection system (3.1) - (3.5). In essence, the performance minimize the kidnappers and abducted subpopulations, as well as the cost of implementation of controls u_1, u_2, therefore the objective functional subject to (4.1) - (4.5) is defined by

$$J(u_1, u_2) = \min \int_0^{t_f} \left[B_1 K + B_2 A + B_3 \frac{u_1^2(t)}{2} + B_4 \frac{u_2^2(t)}{2} \right] dt \tag{4.6}$$

where $B_i \geqslant 0,\ i = 1, 2, \ldots, 4$ denote the weight parameters that balance the size of the terms.

4.1. *Existence of Control*

To show the existence of control problem by the following theorem.

Theorem 4.1. *There exist an optimal control $u^* = (u_1^*, u_2^*)$ of the system (4.1) - (4.5) such that*

$$J(u_1^*, u_2^*) = \min\{J(u_1, u_2) : \ u_1, u_2 \in U\}$$

Proof

The proof follows from the fact that since:

- The set of admissible control $U = \{u_i : u_i \in [0,\ 1]\ Lebesgue\ measurable,\ i = 1, 2\}$ is closed and convex, which implies compactness
- the control dynamics (4.1) - (4.5) is Lipschitz continous and bounded, which determines the compactness for the existence of the optimal control,
- the integrand (Lagrangian) in the objective functional

$$\mathcal{L} = B_1 K + B_2 A + B_3 \frac{u_1^2(t)}{2} + B_4 \frac{u_2^2(t)}{2} \tag{4.7}$$

  is convex on U,

then by the hypothesis of Fillipov existence theorem [49], the control system (4.1) - (4.5) has an optimal solution .

4.2. *Characterization*

To seek for optimal solution (x^*, u^*) to control problem (4.1) - (4.5) using theorem 2.15, define Hamltonian

$$\begin{aligned} H = {} & p_1 \left[\theta\lambda^\alpha + \rho^\alpha C - \frac{b_1^\alpha K}{1 + aK^2} V - \mu^\alpha V \right] + p_2 \left[\frac{b_1^\alpha K}{1 + aK^2} V - [\mu^\alpha + \delta_1^\alpha + \gamma^\alpha (1 + u_1)] K \right] \\ & + p_3 [\gamma^\alpha (1 + u_1) K - (\mu^\alpha + \rho^\alpha) C] + p_4 \left[(1 - \theta)\lambda^\alpha + \tau^\alpha (1 + u_2) A - \frac{b_2^\alpha K}{1 + aK^2} S - \mu^\alpha S \right] \\ & + p_5 \left[\frac{b_2^\alpha K}{1 + aK^2} S - [\mu^\alpha + \delta_2^\alpha + \tau^\alpha (1 + u_2)] A \right] + B_1 K + B_2 A + B_3 \frac{u_1^2(t)}{2} + B_4 \frac{u_2^2(t)}{2} \end{aligned} \tag{4.8}$$

where $p = (p_1, p_2, p_3, p_4, p_5)^T$ is non-trivial vector. Since there is no constraint subject to control dynamics (4.1) - (4.5), the null Hamiltonian gradient $\partial H_2(x^*(t), u^*(t), p(t), p^0) = 0$ implies

$$u_1^*(t) = \min\left\{\max\left[0, \frac{\gamma^\alpha(p_2 - p_3)K}{B_3}\right], 1\right\}, \tag{4.9}$$

$$u_2^*(t) = \min\left\{\max\left[0, \frac{\tau^\alpha(p_5 - p_4)A}{B_4}\right], 1\right\}, \tag{4.10}$$

while the adjoint (or extremal) equation $D^\alpha_{b-}[p](t) = \partial H_1(x^*(t), u^*(t), p(t), p^0)$ gives

$$_tD^\alpha_{t_f} p_1(t) = -\left[\frac{b_1^\alpha K}{1 + aK^2} + \mu^\alpha\right] p_1 + \frac{b_1^\alpha K}{1 + aK^2} p_2, \tag{4.11}$$

$$\begin{aligned} {}_tD^\alpha_{t_f} p_2(t) &= -\frac{b_1^\alpha (1 - aK^2)V}{(1 + aK^2)^2} p_1 + \left[\frac{b_1^\alpha (1 - aK^2)V}{(1 + aK^2)^2} - \mu^\alpha - \delta_1^\alpha - \gamma^\alpha(1 + u_1)\right] p_2 + \gamma^\alpha(1 + u_1)p_3 \\ &\quad - -\frac{b_2^\alpha (1 - aK^2)S}{(1 + aK^2)^2} p_4 + \frac{b_2^\alpha (1 - aK^2)S}{(1 + aK^2)^2} p_5 + B_1, \end{aligned} \tag{4.12}$$

$$_tD^\alpha_{t_f} p_3(t) = \rho^\alpha p_1 - (\mu^\alpha + \rho^\alpha)p_3, \tag{4.13}$$

$$_tD^\alpha_{t_f} p_4(t) = -\left[\frac{b_2^\alpha K}{1 + aK^2} + \mu^\alpha\right] p_4 + \frac{b_2^\alpha K}{1 + aK^2} p_5, \tag{4.14}$$

$$_tD^\alpha_{t_f} p_5(t) = \tau^\alpha(1 + u_2)p_4 - [\mu^\alpha + \delta_2^\alpha + \tau^\alpha(1 + u_2)]p_5, \tag{4.15}$$

where as, the transversality condition

$$_tD^{\alpha-1}_{t_f-} p_i|_{t_f} = 0, \quad \Longrightarrow p_i(t_f) = 0, \quad i = 1, 2, \ldots, 5. \tag{4.16}$$

5. Numerical Simulation

Based on the reported data [15, 13, 16, 50], the associated model paramers are estimated as $\lambda = 9.77 \times 10^{-5}, b_1 = 3.1 \times 10^{-5}, b_2 = 2.5 \times 10^{-4}, \delta_1 = 3.41 \times 10^{-9}, \delta_2 = 9.21 \times 10^{-9}, \gamma = 1.39 \times 10^{-5}, \tau = 6.33 \times 10^{-4}, \mu = 2.96 \times 10^{-5}$. Thus the simulations are run on Matlab using FDE12 code and the following results are obtained.

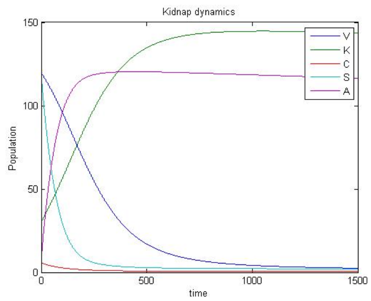

Figure 2: Evolution of population when $a = 0$

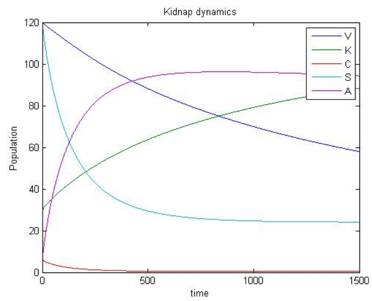

Figure 3: Evolution of population when $a = 10^{-3}$

The efficacy of community vigilance can be seen clearly from figure 2 and 3. They are indicating that at startup of the vigilance with the efficacy $(a = 0)$, the population dynamics persist as usual. However, as the vigillance progress with little efficacy ($a = 10^{-3}$) reducing contact with kidnappers thereby preventing recruitment and abduction, the kidnappers and abducted population begin to decline respctively. Moreover, figures

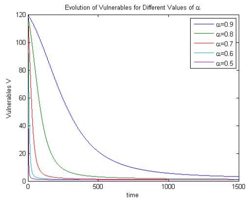

Figure 4: Evolution of Vulnerables for different values of α

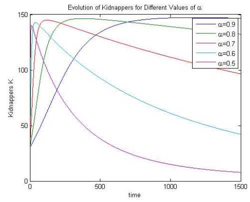

Figure 5: Evolution of kidnapper for different values of α

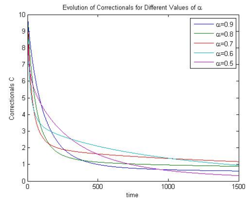

Figure 6: Evolution of correctionals for different values of α

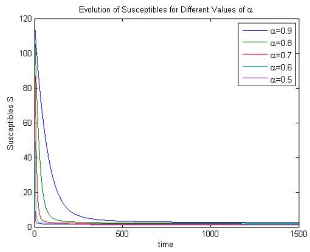

Figure 7: Evolution of susceptibles for different values of α

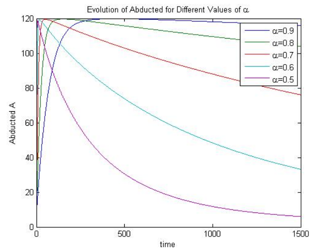

Figure 8: Evolution of abducted for different values of α

4 - 8 depict the individual subpopulations for varrying fractional order (α), which entails the memory effect of fractional derivative operator. Meanwhile, at every time instant one can determine the evolution of the dynamics to know the population changes due to the menace. This further addressed Gottfried Leibnitz's question to L'Hopital in 1695 [51], that asked what would be a differential having as its exponent a fraction for instance $\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}}$?

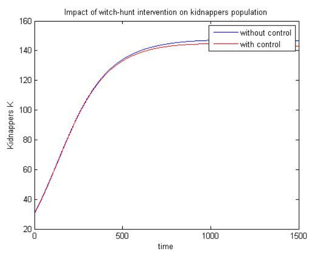

Figure 9: Impact of witch-hunt intervention on kidnappers population

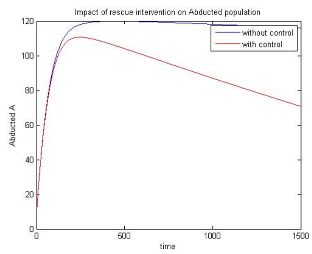

Figure 10: Impact of rescue intervention on Abducted population

The impact of control interventions to witch-hunt and rescue can be seen from figure 9 and 10 respectively, as the dynamics of kidnappers and abducted confinement decline upon implementation.

6. Conclusion

This paper presents fractional order mathematical model describing kidnap coinfection dynamics with recruitment and abduction occurring simultaneously. Due to the role of volunteer vigilantee in reducing mingling, community vigilance was incorporated to assess the efficacy. Subsequent to establishing well-posedness for the positivity and boundedness, the two feasible equilibria (kidnap - free and persistant) of the model are found to be globally asymptotically stable if kidnap propagation parameter R_k is respectively less or bigger than unity. To minize the menace with a minimim cost of security and implementation, optimal control strategies to witch-hunt kidnappers (in their hideout) and rescue the abducted are found to be effective to bring the menace to no avail.

Recommendation

Terrorist groups target young people and their ideology spread in the same pattern as diseases. They exploit real or perceived grievances and use manipulative messages, including through new technologies to increase their reach across borders and cultures. Tight security watch on terror financing made them resort to kidnapping for ransom as a source of funding to sustain their activities. In light of the findings in this research, the following recommendations are made:

1. government should create more job opportunities to engage youth from being recruited by terror sponsors,
2. community policing should be introduced in collaboration with community members, traditional and religious leaders,
3. weapons possession by unauthorized individual is rampant, it should be collected and decisive action should be taken on guilty of an offence,
4. there is urgent need to revive nomadic and adult literacy education system to reach out herder's hamlets and settlements.
5. recruitment capacity of security agencies should be multiplied with well-equipped military hardware for counterterrorism operation,
6. in addition, incentives should be offer to frontline workers for encouragement.

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