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An Analysis of Periodic Motion Using Fractional Calculus

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Abstract

Fractional calculus has gained significant attention from engineers because of its ability to generalise the concept of derivatives to non-integer orders. This study explores the applications of fractional calculus in engineering mathematics, particularly focusing on the analysis of periodic motion. Although extensive research has been conducted in this domain, the proposed models and algorithms are still in their early stages of development. This study examines the harmonic oscillator problem using a fractional derivative damping term, which is proportional to the velocity, instead of the conventional damping term. This paper presents a series of solutions comparing fractional-order solutions and damping ratios, not only for semi-derivatives but also for a range of fractional orders. An association between the fractional order (α) and damping ratio (η) has been elucidated to minimise the computational duration necessary for resolving the fractional equation of motion pertaining to a one-dimensional simple harmonic oscillator. The roots obtained using this method can be applied to solve the simple harmonic oscillations of a mass between two springs with transverse oscillations. This investigation's outcomes advance our understanding of fractional harmonic oscillator behaviour and highlight the efficacy of fractional calculus in tackling intricate engineering challenges.

Keywords: Fractinal Differential Equations, Simple Harmonic Oscillator, Damping.
2010 MSC: 34A08, 34C23, 70K20, 70K20 .

1. Introduction

Fractional calculus (FC) has garnered considerable attention owing to its elegant extension of derivatives to non-integer orders. In recent decades, the engineering community has exhibited growing fascination with the development of various facets of fractional calculus for diverse engineering applications.

In recent years, the field of FC, this mathematical framework expands upon traditional calculus, incorporating fractional orders of integration and differentiation, has attracted substantial scholarly interest owing to its capacity for modelling intricate phenomena

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across various scientific and engineering disciplines [1], [3]. This sophisticated mathematical framework has demonstrated its utility in a wide array of domains, including image processing, computer vision, mechanics, and the study of nanoscale flows [1], [4], [5].

Fractional calculus offers a novel approach for analysing and describing the behaviour of oscillating systems in the context of periodic motion. The application of fractional calculus to periodic functions presents intriguing properties and challenges. While fractional derivatives of periodic signals defined on the entire real line maintain periodicity, causal periodic signals lose their periodic nature under fractional differentiation [2]. This distinction highlights the importance of carefully considering domain and causality when applying fractional operators to periodic systems. Moreover, research into fractional linear systems has revealed that sinusoidal impulse responses are exclusive to integer-order linear systems, Examining the distinctive characteristics of fractional order systems in the context of periodic motion investigation [2].

The integration of FC into the study of periodic motion opens new avenues for research and applications. From the analysis of projectile motion using fractional differential equations [6] to the development of fractional Euler-Lagrange equations for variational problems [7], FC provides a robust analytical framework for investigating the intricacies of periodic phenomena. As the field continues to evolve, the combination of fractional calculus and periodic motion analysis promises to yield valuable insights and innovative approaches in various scientific and engineering fields.

Extensive historical overviews of fractional calculus can be found in [17] and [14]. Significant research efforts have been dedicated to exploring damping and viscoelasticity in dynamic systems. [16] introduced a numerical approach for resolving a fractionally damped spring mass damper system with a single degree of freedom, examining a derivative of order 0.5 and utilising the Laguerre integral formula. A comparative analysis of numerical methods is presented in [9]. Implementing a FC approach with a 0.5-order derivative, in conjunction with an eigenvector decomposition technique, [10] provided a closed-form solution to the problem. The issue was addressed using the Laplace transform in [12], whilst [11] exclusively applied the Fourier transform with a fractional order of 0.5. [15] This study investigated the initialisation challenge associated with a system of linear fractional-order differential equations, specifically addressing discrete values of the fractional order of derivation. The research methodology incorporated the application of the Mittag-Leffler function to analyse this complex mathematical problem. In order to illustrate the gradual transition from solid to fluid state as the memory parameter ranges from zero to one, [13] examined the relaxation functions and creep by modifying the derivation order across a spectrum of 0.05 to 0.35. These results were expressed using the Mittag-Leffler function. A comprehensive review and comparison of various numerical solutions to fractional equations was provided by [8].

This study extends the analysis by exploring the implications of the alternative damping term on the behaviour of a system across a broader range of fractional orders. The investigation also delves into the physical interpretation of the fractional derivative term and its impact on the energy dissipation in the oscillator system. Furthermore, the established correlation between α and β opens up possibilities for optimising the numerical solution process and developing more efficient computational methods for similar fractional differential equations.

2. Definitions and Preliminaries

2.1. Periodic motion:

It is well known that a motion repeated at regular intervals is known as periodic motion. This period is the name given to these uniform gaps or cycles of motion. Frequency represents the number of periods in one second.

2.2. Simple Harmonic motion (SHM):

Periodic motion is a specific case of SHM wherein the magnitude of the object's equilibrium position is directly inversely proportional to the restoring force on the moving object. It initiates and sustains oscillation indefinitely provided that friction or any other energy loss is absent. [Wikipedia] In accordance with Hooke's law, a mass experiences a linear elastic restoring force when subjected to displacement, various SHM serve as mathematical models of motion; however, they are characterised by the oscillation of the mass in the spring. The SHM system is called a simple harmonic oscillator (SHO). SHO cannot be driven or dampened. It comprises a m - mass that is subject to a single force (F), which is solely dependent on the x position and acts upon the mass in the direction of $x = 0$. We know that

$$m\ddot{x} + kx = 0 \tag{2.1}$$

$$\omega^2 = \frac{k}{m} \tag{2.2}$$

Where ω represents the particle's angular velocity, and the constant component of the spring, or its stiffness, is denoted by k.

3. Main Result

3.1. Fractional Simple Harmonic Oscillator

The general form of fractional equation of motion for one dimensional simple harmonic oscillation is

$$m\frac{d^2x}{dt^2} + \alpha b\frac{d^\alpha x}{dt^\alpha} + kx = 0 \tag{3.1}$$

where α is non-integer and $\alpha = 0$ corresponds to an undamped simple harmonic oscillator and $\alpha = 1$ corresponds to a damped simple harmonic oscillator.
Then we will start with the ansatz

$$x(t) = e^{\omega t} \tag{3.2}$$

The Solution of the fractional oscillator is reduced to the examination of the zeros of the polynomial

$$\alpha b\omega^\alpha + 2k = 0$$

$$\omega^\alpha + \frac{2k}{\alpha b} = 0 \tag{3.3}$$

We know that

$$\eta = \frac{b}{2m\omega^{2-\alpha}} \tag{3.4}$$

where η is the damping ratio and the power on ω was introduced for consistency of dimensions, we get

$$\alpha(2m\omega^{2-\alpha}\eta)\omega^{\alpha} + 2k = 0$$

$$2m\alpha\eta\omega^2 + 2k = 0$$

$$m\alpha\eta\frac{k}{m} + k = 0$$

$$k(1+\alpha\eta) = 0$$

$$\text{either} \quad k = 0 \quad \text{or} \quad 1+\alpha\eta = 0$$

but

$$k \neq 0$$

$$\therefore 1+\alpha\eta = 0$$

$$\alpha\eta = -1 \tag{3.5}$$

Now we will go with the fractional equation of motion of the one dimensional simple harmonic oscillator in its general form as follows:

$$m\frac{d^2x}{dt^2} + \alpha b\frac{d^{\alpha}x}{dt^{\alpha}} + kx = 0$$

Where α is a non integer and $\alpha = 0$ corresponds to an undamped simple harmonic oscillator. Dividing above equation by m and using

$$\eta = \frac{b}{2m\omega^{2-\alpha}}$$

and

$$\omega^2 = \frac{k}{m}$$

and applying Laplace Transform to the equation we get:

$$L\left[\frac{d^2x}{dt^2}\right] + 2\alpha\eta\omega^{2-\alpha}L\left[\frac{d^{\alpha}x}{dt^{\alpha}}\right] + \omega^2 L(x) = 0$$

$$s^2x(s) - sx_0 - x'(0) + 2\alpha\eta\omega^{2-\alpha}\left[s^{\alpha}x(s) - \sum_{k=0}^{n-1} s^k\frac{d^{\alpha-1-k}x}{dt^{\alpha-1-k}}\right] + \omega^2x(s) = 0$$

$$s^2x(s) - sx_0 + 2\alpha\eta\omega^{2-\alpha}[s^{\alpha}x(s) - c] + \omega^2x(s) = 0$$

where,

$$c = \sum_{k=0}^{n-1} s^k\frac{d^{\alpha-1-k}x}{dt^{\alpha-1-k}}$$

$$x(s)[s^2 + 2\alpha\eta\omega^{2-\alpha}s^{\alpha} + \omega^2] = sx_0 + 2c\alpha\eta\omega^{2-\alpha}$$

Now using the result,

$$\alpha\eta = -1$$

We get,

$$x(s)[s^2 - 2\omega^{2-\alpha}s^\alpha + \omega^2] = sx_0 - 2c\omega^{2-\alpha}$$

$$x(s) = \frac{sx_0 - 2c\omega^{2-\alpha}}{s^2 - 2\omega^{2-\alpha}s^\alpha + \omega^2}$$

$$x(s) = x_1 + x_2$$

Where,

$$x_1 = \frac{sx_0}{s^2 - 2\omega^{2-\alpha}s^\alpha + \omega^2}$$

and

$$x_2 = \frac{-2c\omega^{2-\alpha}}{s^2 - 2\omega^{2-\alpha}s^\alpha + \omega^2}$$

Now

$$x_1 = \frac{sx_0}{s^2 - 2\omega^{2-\alpha}s^\alpha\left[1 + \frac{\omega^2}{s^2 - 2\omega^{2-\alpha}s^\alpha}\right]}$$

$$= \frac{sx_0}{s^2 - 2\omega^{2-\alpha}s^\alpha}\sum_{p=0}^{\infty}(-1)^p\frac{\omega^{2p}}{[s^2 - 2\omega^{2-\alpha}s^\alpha]^p}$$

$$= \frac{sx_0}{s^2\left[1 - \frac{2\omega^{2-\alpha}}{s^{2-\alpha}}\right]}\sum_{p=0}^{\infty}(-1)^p\frac{\omega^{2p}}{s^{2p}\left[1 - \frac{2\omega^{2-\alpha}}{s^2}s^\alpha\right]^p}$$

$$= x_0\sum_{p=0}^{\infty}(-1)^p\frac{\omega^{2p}}{s^{2p+1}\left[1 - \frac{2\omega^{2-\alpha}}{s^{2-\alpha}}\right]^{p+1}}$$

$$= x_0\sum_{p=0}^{\infty}(-1)^p\frac{\omega^{2p}}{s^{2p+1}} * \sum_{r=0}^{\infty}\frac{(p+1)!}{r!(p+1-r)!}\frac{2\omega^{2-\alpha}}{s^{2-\alpha}}$$

$$= x_0\sum_{p=0,r=0}^{\infty}(-1)^p\frac{(p+1)!}{r!(p+1-r)!}\frac{\omega^{2p}}{s^{(2p+1)}} * \left(\frac{2\omega^{(2-\alpha)}}{s^{2-\alpha}}\right)^r$$

$$= x_0\sum_{p=0,r=0}^{\infty}(-1)^p\frac{(p+1)!}{(r!)(p+1-r)!} * 2^r * \frac{\omega^{2p+2r-\alpha r}}{s^{2p+1+2r-\alpha r}}$$

$$\therefore x_1 = x_0\sum_{p,r=0}^{\infty}\frac{(-1)^p(p+1)!2^r\omega^{2p}\omega^{r(2-\alpha)}}{r!(p-r+1)!s^{2(p+r)-\alpha r+1}}$$

Similarly,

$$x_2 = -2c\omega^{2-\alpha}\sum_{p,r=0}^{\infty}\frac{(-1)^p(p+1)!2^r\omega^{2p}\omega^{r(2-\alpha)}}{r!(p-r+1)!s^{2(p+r)-\alpha r+2}}$$

$$\therefore x(s) = x_0\sum_{p,r=0}^{\infty}\frac{(-1)^p(p+1)!2^r\omega^{2p}\omega^{r(2-\alpha)}}{r!(p-r+1)!s^{2(p+r)-\alpha r+1}} - 2c\omega^{2-\alpha}\sum_{p,r=0}^{\infty}\frac{(-1)^p(p+1)!2^r\omega^{2p}\omega^{r(2-\alpha)}}{r!(p-r+1)!s^{2(p+r)-\alpha r+2}}$$

By taking Laplace inverse on both sides, We get,

$$x(t) = x_0 \sum_{p,r=0}^{\infty} \frac{(-1)^p (p+1)! 2^r \omega^{2p} \omega^{r(2-\alpha)}}{r!(p-r+1)!} * \frac{t^{2(p+r)-\alpha r}}{\sqrt{2(p+r)-\alpha r+1}}$$

$$-2c\omega^{2-\alpha} \sum_{p,r=0}^{\infty} \frac{(-1)^p (p+1)! 2^r \omega^{2p} \omega^{r(2-\alpha)}}{r!(p-r+1)!} * \frac{t^{2(p+r)-\alpha r+1}}{\sqrt{2(p+r)-\alpha r+2}}$$

$$\therefore x(t) = \sum_{p,r=0}^{\infty} \frac{(-1)^p (p+1)! 2^r \omega^{2p} \omega^{r(2-\alpha)} t^{2(p+r)-\alpha r}}{r!(p-r+1)!\Gamma(2(p+r)-\alpha r+1)} * \left[x_0 - 2c\omega^{2-\alpha} \frac{t}{2(p+r)-\alpha r+1}\right]$$

Here c can be obtained depending on the problem to be discussed. In the above expansion we will take $c = x_0\omega^{\alpha-1}$ for sake of maintaining the dimensions.
The above expansion is valid for a relatively short time. But we can, using analytic continuation, prolong the validity of an equation similar to the expansion above for a long time.

4. Convergence of above series

In the obtained series solution, if we want to check the convergence, then we just have to check the convergence of following term:

$$\sum_{p,r=0}^{\infty} \frac{t^{2(p+r)-\alpha r}}{\Gamma(2(p+r)-\alpha r+1}$$

$$= \sum_{p,r=0}^{\infty} \frac{t^{2p} * t^{2r-\alpha r}}{\Gamma(2(p+r)-\alpha r+1}$$

$$= \sum_{p,r=0}^{\infty} \frac{t^{2p} * t^{r(2-\alpha)}}{\Gamma(2(p+r)-\alpha r+1}$$

$$= \sum_{p,r=0}^{\infty} \frac{t^{2p} * t^{r(2-\alpha)}}{\frac{\Gamma(2p+1)*\Gamma r(2-\alpha)}{\beta(2p+1, r(2-\alpha))}}$$

$$= \sum_{p,r=0}^{\infty} \beta(2p+1, r(2-\alpha)) \frac{t^{2p} * t^{r(2-\alpha)}}{\Gamma(2p+1)*\Gamma r(2-\alpha)}$$

$$= \sum_{p,r=0}^{\infty} \beta(2p+1, r(2-\alpha)) * \sum_{p=0}^{\infty} \frac{t^{2p}}{\Gamma(2p+1)} * \sum_{r=0}^{\infty} \frac{t^{r(2-\alpha)}}{\Gamma r(2-\alpha)}$$

Here, in $\beta(2p+1, r(2-\alpha))$ both $2p+1$ and $r(2-\alpha)$ are positive.
Hence, the series

$$\sum_{p,r=0}^{\infty} \beta(2p+1, r(2-\alpha))$$

is convergent.
And

$$\sum_{p=0}^{\infty} \frac{t^{2p}}{\Gamma(2p+1)},$$

$$\sum_{r=0}^{\infty} \frac{t^{r(2-\alpha)}}{\Gamma r(2-\alpha)}$$

both represent a Mittag-Leffler function which is a convergent series.
So, the series obtained as a result is a convergent series.

5. Conclusion

This study demonstrates the potential of fractional calculus for analysing periodic motion and solving complex engineering problems. By examining the harmonic oscillator problem with a fractional derivative damping term, In this study, we obtained valuable insights into the behaviour of fractional harmonic oscillators. The established relationship between the fractional order (α) and damping ratio (η) offers a more efficient approach for solving the fractional equation of motion for a simple one-dimensional harmonic oscillator. These findings not only contribute to the advancement of engineering mathematics but also pave the way for future applications in areas such as the analysis of transverse oscillations in mass-spring systems. As the field of fractional calculus continues to evolve, further research in this domain promises to yield innovative solutions and deepen our understanding of the complex physical phenomena.

6. Conflict of Interest

The researchers have affirmed the absence of any conflicts of interest pertaining to this study.

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