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Existence Results for Implicit Fractional Differential Equations with Riesz-Caputo Derivative

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Abstract

This article investigates the existence and uniqueness of solutions for a class of initial value problems involving implicit fractional differential equations with the Riesz–Caputo fractional derivative. By employing fixed point theorems in conjunction with the technique of measures of noncompactness, we establish key existence and uniqueness results. Furthermore, we demonstrate that the proposed problem exhibits Ulam stability. To support and illustrate our theoretical findings, several examples are provided.

Keywords: Riesz-Caputo fractional derivative, implicit problem, existence, measure of noncompactness, fixed point, Ulam stability.
2010 MSC: 26A33, 34A08.

1. Introduction

Recently, fractional calculus has been a very useful tool in modeling of many phenomena in applications and sciences, such as physics, engineering, electrochemistry, geology, stability, controllability and signal theory, and many other fields. For more details, see [1, 2, 3, 4, 6, 8, 15, 16, 21, 24, 25, 26, 27, 28, 30, 29] and the references therein.

There are numerous fractional derivatives, each with its own set of characteristics and uses. The Riemann-Liouville fractional derivative, introduced in 1847, and the Caputo derivative, created later in 1967, are two notable examples. Among the other notable derivatives are the Hilfer derivative (2000), the Hadamard derivative (1892), and the

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Caputo-Fabrizio derivative (2015). In many instances, the current condition of a process is determined by its past and future evolution. Stock price options, for example, depend on forecasting future market patterns. Similarly, fractional derivatives are used to describe the concentration of diffusion on a specific route in the anomalous diffusion problem. The Riesz derivative, a two-sided fractional operator, is especially helpful in this situation because it can capture both past and future memory effects. This property is particularly useful when describing fractional processes on a finite area. The Riemann-Liouville and Caputo fractional derivatives, which are one-sided fractional operators that only reflect past or future memory effects, are currently the center of much work on fractional differential equations. The flexibility of the Riesz derivative, on the other hand, has attracted notice and is garnering favor in the field. For further information, interested readers may refer to the works cited in [9, 10, 12].

In many cases, determining the exact solution of differential equations is difficult, if not unattainable. It is usual in such situations to investigate approximate solutions. It is essential to observe, however, that only steady approximations are accepted. As a consequence, different stability analysis techniques are used. S. M. Ulam, a mathematician, first raised the stability issue in functional equations in a 1940 lecture at Wisconsin University. In his presentation, Ulam posed the following challenge: "Under what conditions does an additive mapping exist near an approximately additive mapping?" [32]. The following year, Hyers provided an answer to Ulam's problem for additive functions defined on Banach spaces [13]. In 1978, Rassias further expanded upon Hyers' work, demonstrating the existence of unique linear mappings near approximate additive mappings [22]. Since then, numerous research articles in the literature have addressed the stabilities of various types of differential and integral equations. Interested readers may refer to [18, 28, 16] and their respective references for further details.

The authors of [9] studied the existence of solution for the following boundary value problem:

$$\begin{cases} {}_0^{RC}D_{\varkappa}^{\alpha} y(\theta) = g(\theta, y(\theta)), \quad \theta \in \Theta := [0, \varkappa], \\ y(0) = y_0, \quad y(\varkappa) = y_{\varkappa}, \end{cases}$$

where ${}_0^{RC}D_{\varkappa}^{\alpha}$ is a Riesz-Caputo derivative of order $0 < \alpha \leqslant 1$, $g : \Theta \times \mathbb{R} \to \mathbb{R}$ a continuous function and $y_0 \in \mathbb{R}$. Their arguments are based on Leray-Schauder fixed point theorem, and Schauder fixed point theorem.

In [17], Li and Wang discussed the following fractional problem:

$$\begin{aligned} {}_0^{RC}D_1^{\gamma} y(t) = f(t, y(t)), \quad t \in [0, 1], \quad 0 < \gamma \leqslant 1, \\ y(0) = a, \quad y(1) = by(\eta), \end{aligned}$$

where ${}_0^{RC}D_1^{\gamma}$ is the Riesz Caputo derivative, $f \in C([0, 1] \times [0, +\infty), [0, +\infty)), 0 < \eta < 1, a > 0, 0 < b < 2$. They found the positive solutions by applying the technique of monotone iterative.

Naas *et al.* [20] investigated the existence and uniqueness results of the following fractional differential equation with the Riesz-Caputo derivative:

$$\begin{cases} {}_0^{RC}D_T^{\vartheta}\varkappa(t) + \mathfrak{F}\left(t, \varkappa(t), {}_0^{RC}D_T^{\sigma}\varkappa(t)\right) = 0, t \in \mathcal{J} := [0, T], \\ \varkappa(0) + \varkappa(T) = 0, \quad \mu\varkappa'(0) + \sigma\varkappa'(T) = 0, \end{cases}$$

where $1 < \vartheta \leqslant 2$ and $, 0 < \sigma \leqslant 1, {}_0^{RC}D_T^{\kappa}$ is the Riesz-Caputo fractional derivative of order $\kappa \in \{\vartheta, \sigma\}$, $\mathfrak{F} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is a continuous function, and μ, σ are nonnegative constants with $\mu > \sigma$. The existence and uniqueness of solutions for their problem are demonstrated with the Riesz-Caputo derivatives via Banach's, Schaefer's, and Krasnoselskii's fixed point theorems.

Existence of the solution for implicit initial value problem is one of the important topics of fractional differential equations. In this paper, we present some existence results for the following implicit fractional problem:

$$ {}_0^{RC}D_{\varkappa}^{\alpha}y(\theta) = \varphi\left(\theta, y(\theta), {}_0^{RC}D_{\varkappa}^{\alpha}y(\theta)\right), \quad \theta \in \Theta := [0, \varkappa], \tag{1.1}$$

$$ y(0) = y_0, \tag{1.2}$$

where ${}_0^{RC}D_{\varkappa}^{\alpha}$ represent the Riesz-Caputo derivative of order $0 < \alpha \leqslant 1$, $\varphi : \Theta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $y_0 \in \mathbb{R}$.

We also establish the Ulam stability for the same problem but in Banach spaces. Finally, some examples are given to illustrate the applications of the main results.

2. Preliminaries

In this section, we introduce some notations, definitions, and preliminary facts which are used throughout this paper.

We denote by $C(\Theta, E)$ the Banach space of all continuous functions from Θ to E, with the norm

$$\|\xi\|_{\infty} = \sup\{\|\xi(\theta)\| : \theta \in \Theta\}.$$

Definition 2.1 ([14]). Let $\alpha > 0$. The left and right Riemann-Liouville fractional integrals of a function $\varphi \in C(\Theta, E)$ of order α are given respectively by

$${}_0I_{\theta}^{\alpha}\varphi(\theta) = \frac{1}{\Gamma(\alpha)}\int_0^{\theta} (\theta - \rho)^{\alpha-1}\varphi(\rho)d\rho,$$

and

$${}_{\theta}I_{\varkappa}^{\alpha}\varphi(\theta) = \frac{1}{\Gamma(\alpha)}\int_{\theta}^{\varkappa} (\rho - \theta)^{\alpha-1}\varphi(\rho)d\rho,$$

where $\Gamma(.)$ is the Gamma function defined by

$$\Gamma(\omega) = \int_0^{\infty} \rho^{\omega-1}e^{-\rho}d\rho, \quad \omega > 0.$$

Definition 2.2 ([14]). Let $\alpha > 0$. The Riesz fractional integral of a function $\varphi \in C(\Theta, E)$ of order α is defined by

$$\begin{aligned} {}_0I^{\alpha}_{\varkappa}\varphi(\theta) &= \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa}|\theta-\rho|^{\alpha-1}\varphi(\rho)d\rho \\ &= {}_0I^{\alpha}_{\theta}\varphi(\theta) + {}_\theta I^{\alpha}_{\varkappa}\varphi(\theta), \end{aligned}$$

where ${}_0I^{\alpha}_{\theta}$ and ${}_\theta I^{\alpha}_{\varkappa}$ are the left and right fractional integrals of Riemann-Liouville.

Definition 2.3 ([14]). Let $\alpha \in (n, n+1]$, $n \in \mathbb{N}_0$. The left and right Caputo fractional derivatives of a function $\varphi \in C^{n+1}(\Theta, E)$ of order α are given respectively by

$${}^C_0D^{\alpha}_{\theta}\varphi(\theta) = \frac{1}{\Gamma(n+1-\alpha)}\int_0^{\theta}(\theta-\rho)^{n-\alpha}\varphi^{(n+1)}(\rho)d\rho,$$

and

$${}^C_\theta D^{\alpha}_{\varkappa}\varphi(\theta) = \frac{(-1)^{n+1}}{\Gamma(n+1-\alpha)}\int_\theta^{\varkappa}(\rho-\theta)^{n-\alpha}\varphi^{(n+1)}(\rho)d\rho.$$

Definition 2.4 ([14]). Let $\alpha \in (n, n+1]$, $n \in \mathbb{N}_0$. The Riesz-Caputo fractional derivative of a function $\varphi \in C^{n+1}(\Theta, E)$ of order α is given by

$$\begin{aligned} {}^{RC}_0D^{\alpha}_{\varkappa}\varphi(\theta) &= \frac{1}{\Gamma(n+1-\alpha)}\int_0^{\varkappa}|\theta-\rho|^{n-\alpha}\varphi^{(n+1)}(\rho)d\rho \\ &= \frac{1}{2}({}^C_0D^{\alpha}_{\theta}\varphi(\theta) + (-1)^{n+1}{}^C_\theta D^{\alpha}_{\varkappa}\varphi(\theta)), \end{aligned}$$

where ${}^C_0D^{\alpha}_{\theta}$ is the left Caputo derivative and ${}^C_\theta D^{\alpha}_{\varkappa}$ is the right one. If we take $0 < \alpha \leqslant 1$ and $\varphi \in C(\Theta, E)$, we obtain

$${}^{RC}_0D^{\alpha}_{\varkappa}\varphi(\theta) = \frac{1}{2}({}^C_0D^{\alpha}_{\theta}\varphi(\theta) - {}^C_\theta D^{\alpha}_{\varkappa}\varphi(\theta)).$$

Lemma 2.5 ([14]). *If $\xi \in C^{n+1}(\Theta, E)$ and $\alpha \in (n, n+1]$, $n \in \mathbb{N}_0$, then we have*

$${}_0I^{\alpha}_{\theta}\,{}^C_0D^{\alpha}_{\theta}\xi(\theta) = \xi(\theta) - \sum_{k=0}^{n}\frac{\xi^{(k)}(0)}{k!}(\theta-0)^k,$$

and

$${}_\theta I^{\alpha}_{\varkappa}\,{}^C_\theta D^{\alpha}_{\varkappa}\xi(\theta) = (-1)^{n+1}\left[\xi(\theta) - \sum_{k=0}^{n}\frac{(-1)^k\xi^{(k)}(\varkappa)}{k!}(\varkappa-\theta)^k\right].$$

Consequently, we may have

$${}_0I^{\alpha}_{\varkappa}\,{}^{RC}_0D^{\alpha}_{\varkappa}\xi(\theta) = \frac{1}{2}({}_0I^{\alpha}_{\theta}\,{}^C_0D^{\alpha}_{\theta}\xi(\theta) + (-1)^{n+1}\,{}_\theta I^{\alpha}_{\varkappa}\,{}^C_\theta D^{\alpha}_{\varkappa}\xi(\theta)).$$

In particular, if $0 < \alpha \leqslant 1$, then we obtain

$${}_0I^{\alpha}_{\varkappa}\,{}^{RC}_0D^{\alpha}_{\varkappa}\xi(\theta) = \xi(\theta) - \frac{1}{2}(\xi(0) + \xi(\varkappa)).$$

2.1. Measure of Noncompactness

Definition 2.6 ([7]). Let Ξ be a Banach space and let Ψ_Ξ be the family of bounded subsets of Ξ. The Kuratowski measure of noncompactness is the map $\zeta : \Psi_\Xi \longrightarrow [0, \infty)$ defined by

$$\zeta(\chi) = \inf \left\{ \varepsilon > 0 : \chi \subset \bigcup_{j=1}^{m} \chi_j, \mathrm{diam}(\chi_j) \leqslant \varepsilon \right\},$$

where $\chi \in \Psi_\Xi$.

The map ζ satisfies the following properties:

- $\zeta(\chi) = 0 \Leftrightarrow \overline{\chi}$ is compact (χ is relatively compact).
- $\zeta(\chi) = \zeta(\overline{\chi})$.
- $\chi_1 \subset \chi_2 \Rightarrow \zeta(\chi_1) \leqslant \zeta(\chi_2)$.
- $\zeta(\chi_1 + \chi_2) \leqslant \zeta(\Omega_1) + \zeta(\Omega_2)$.
- $\zeta(c\chi) = |c|\zeta(\chi)$, $c \in$.
- $\zeta(\mathrm{conv}\chi) = \zeta(\chi)$.

Lemma 2.7 ([11]). *Let $\Omega \subset C(\Theta, E)$ be a bounded and equicontinuous set. Then*

a) *The function $\theta \to \zeta(\Omega(\theta))$ is continuous on Θ, and*

$$\zeta_C(\Omega) =_{\theta \in J} \zeta(\Omega(\theta)).$$

b) $\zeta\left(\int_0^{\varkappa} \xi(\rho)d\rho : \xi \in \Omega\right) \leqslant \int_0^{\varkappa} \zeta(\Omega(\rho))d\rho$, *where*

$$\Omega(\theta) = \{\xi(\theta) : \xi \in \Omega\}, \theta \in \Theta.$$

2.2. Some Fixed Point Theorems

Theorem 2.8 (Banach's fixed point theorem [31]). *Let E be a Banach space and $\mathcal{H} : E \longrightarrow E$ a contraction, i.e. there exists $k \in [0, 1)$ such that*

$$\|\mathcal{H}(\xi_1) - \mathcal{H}(\xi_2)\| \leqslant k\|\xi_1 - \xi_2\|, \qquad \text{for all } \xi_1, \xi_2 \in E.$$

Then $\mathcal{H}$ has a unique fixed point.

Theorem 2.9 (Schauder's fixed point theorem [31]). *Let E be a Banach space, D a bounded, closed, convex subset of E, and $\mathcal{H} : D \longrightarrow D$ a compact and continuous map. Then $\mathcal{H}$ has at least one fixed point in D.*

Theorem 2.10 (Mönch's fixed point theorem [19]). *Let D be a non-empty, closed, bounded and convex subset of a Banach space E such that $0 \in D$ and let $\mathcal{H} : D \longrightarrow D$ be a continuous mapping. If the implication*

$$\Omega = \overline{\mathrm{conv}}\mathcal{H}(\Omega) \text{ or } \Omega = \mathcal{H}(\Omega) \cup \{0\} \Rightarrow \alpha(\Omega) = 0,$$

holds for every subset Ω of D, then $\mathcal{H}$ has at least one fixed point.

3. Existence Results

We consider the following fractional differential problem:

$$ {}_{0}^{RC}D_{\varkappa}^{\alpha}y(\theta) = \varpi(\theta), \qquad 0 < \alpha \leqslant 1, \quad \theta \in \Theta, \tag{3.1} $$

$$ y(0) = y_0 \in \mathbb{R}, \tag{3.2} $$

where $\varpi : \Theta \to$ is a continuous function.

Lemma 3.1. *The problem* (3.1)-(3.2) *has a unique solution, which is*

$$ y(\theta) \ = \ y_0 - \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} \rho^{\alpha-1}\varpi(\rho)d\rho + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} |\theta-\rho|^{\alpha-1}\varpi(\rho)d\rho. \tag{3.3} $$

Proof. From Definition 2.2, Definition 2.4, and Lemma 2.5, we have

$$ {}_{0}I_{\varkappa}^{\alpha}\, {}_{0}^{RC}D_{\varkappa}^{\alpha}y(\theta) = y(\theta) - \frac{1}{2}(y(0)+y(\varkappa)), $$

which implies that

$$ \begin{aligned} y(\theta) &= \frac{1}{2}(y(0)+y(\varkappa)) + {}_{0}I_{\varkappa}^{\alpha}\varpi(\theta), \\ &= \frac{1}{2}(y(0)+y(\varkappa)) + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} |\theta-\rho|^{\alpha-1}\varpi(\rho)d\rho \\ &= \frac{1}{2}(y(0)+y(\varkappa)) + \frac{1}{\Gamma(\alpha)} \int_0^{\theta} (\theta-\rho)^{\alpha-1}\varpi(\rho)d\rho + \frac{1}{\Gamma(\alpha)} \int_{\theta}^{\varkappa} (\rho-\theta)^{\alpha-1}\varpi(\rho)d\rho. \end{aligned} $$

For $\theta = 0$, we have

$$ y(\varkappa) = y_0 - \frac{2}{\Gamma(\alpha)} \int_0^{\varkappa} \rho^{\alpha-1}\varpi(\rho)d\rho. $$

Then, the final solution is given by:

$$ y(\theta) \ = \ y_0 - \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} \rho^{\alpha-1}\varpi(\rho)d\rho + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} |\theta-\rho|^{\alpha-1}\varpi(\rho)d\rho. $$

□

Conversely, we can easly show by Lemma 2.5, that if y satisfies (3.3), then it satisfies the equation (3.1) and the condition (3.2).

Definition 3.2. By a solution of problem (1.1)-(1.2) we mean a function $y \in C(\Theta, \mathbb{R})$ that satisfies the equation (1.1) and the condition (1.2).

Lemma 3.3. *Let $\varphi : \Theta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, the problem* (1.1)-(1.2) *is equivalent to the following integral equation:*

$$ y(\theta) = y_0 - \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} \rho^{\alpha-1}\varphi(\rho, y(\rho), \varpi_y(\rho))d\rho + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} |\theta-\rho|^{\alpha-1}\varphi(\rho, y(\rho), \varpi_y(\rho))d\rho, $$

where $\varpi_y \in C(\Theta, \mathbb{R})$ satisfies the following functional equation

$$ \varpi_y(\theta) = \varphi(\theta, y(\theta), \varpi_y(\theta)). $$

We are now in a position to prove the existence result of the problem (1.1)-(1.2) based on the Banach's contraction principle.

Let us assume the following assumptions:

$(A1)$ The function $\varphi : \Theta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

$(A2)$ There exist constants $\psi_1 > 0$ and $0 < \psi_2 < 1$ such that

$$|\varphi(\theta, \xi, \eta) - \varphi(\theta, \bar{\xi}, \bar{\eta})| \leqslant \psi_1 |\xi - \bar{\xi}| + \psi_2 |\eta - \bar{\eta}|,$$

for any $\xi, \eta, \bar{\xi}, \bar{\eta} \in \mathbb{R}$ and $\theta \in \Theta$.

Theorem 3.4. *Assume that the assumptions* (A1)-(A2) *hold. If*

$$\frac{2\psi_1 \varkappa^{\alpha}}{(1-\psi_2)\Gamma(\alpha+1)} < 1, \tag{3.4}$$

then the implicit fractional problem (1.1)-(1.2) *has a unique solution on* Θ.

Proof. Let us transform the problem (1.1)-(1.2) into a fixed point problem by defining the operator $\aleph : C(\Theta, \mathbb{R}) \longrightarrow C(\Theta, \mathbb{R})$ by:

$$\begin{aligned} \aleph y(\theta) = y_0 &- \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} \rho^{\alpha-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} |\theta - \rho|^{\alpha-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho. \end{aligned}$$

Obviously, the fixed points of the operator $\aleph$ are solutions of the problem (1.1)-(1.2). Let $y, z \in C(\Theta, \mathbb{R})$ and $\theta \in \Theta$. Then, we have

$$\begin{aligned} |\aleph y(\theta) - \aleph z(\theta)| \leqslant &\frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} \rho^{\alpha-1} |\varphi(\rho, y(\rho), \varpi_y(\rho)) - \varphi(\rho, z(\rho), \varpi_z(\rho))| d\rho \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} |\theta - \rho|^{\alpha-1} |\varphi(\rho, y(\rho), \varpi_y(\rho)) - \varphi(\rho, z(\rho), \varpi_z(\rho))| d\rho, \end{aligned}$$

where

$$\varpi_y(\theta) = \varphi(\theta, y(\theta), \varpi_y(\theta)),$$

and

$$\varpi_z(\theta) = \varphi(\theta, z(\theta), \varpi_z(\theta)).$$

Then, by $(A2)$ we find that

$$\begin{aligned} |\varpi_y(\theta) - \varpi_z(\theta)| &= |\varphi(\theta, y(\theta), \varpi_y(\theta)) - \varphi(\theta, z(\theta), \varpi_z(\theta))| \\ &\leqslant \psi_1 |y(\theta) - z(\theta)| + \psi_2 |\varpi_y(\theta) - \varpi_z(\theta)|, \end{aligned}$$

which implies

$$|\varpi_y(\theta) - \varpi_z(\theta)| \leqslant \frac{\psi_1}{1-\psi_2} |y(\theta) - z(\theta)|.$$

As a consequence, we get

$$\begin{aligned}|\aleph y(\theta)-\aleph z(\theta)| &\leqslant \frac{\psi_1}{(1-\psi_2)\Gamma(\alpha)}\|y-z\|_\infty\int_0^{\varkappa}\rho^{\alpha-1}d\rho \\ &\quad+\frac{\psi_1}{(1-\psi_2)\Gamma(\alpha)}\|y-z\|_\infty\int_0^{\varkappa}|\theta-\rho|^{\alpha-1}d\rho \\ &\leqslant \frac{2\psi_1\varkappa^\alpha}{(1-\psi_2)\Gamma(\alpha+1)}\|y-z\|_\infty.\end{aligned}$$

Thus,

$$\|\aleph y-\aleph z\|_\infty \leqslant \frac{2\psi_1\varkappa^\alpha}{(1-\psi_2)\Gamma(\alpha+1)}\|y-z\|_\infty.$$

Consequently, by the Banach's contraction principle, the operator $\aleph$ has a unique fixed point which is solution of the problem (1.1)-(1.2). □

Remark 3.5. Let us put

$$q_1(\theta)=|\varphi(\theta,0,0)|,\ \psi_1=q_2^*,\ \psi_2=q_3^*.$$

Then, the hypothesis $(A2)$ implies that

$$|\varphi(\theta,\xi,\eta)|\leqslant q_1(\theta)+q_2^*|\xi|+q_3^*|\eta|,$$

for $\theta\in\Theta$, $\xi,\eta\in\mathbb{R}$ and $q_1\in C(\Theta,\mathbb{R}_+)$, with

$$q_1^* =_{\theta\in\Theta} q_1(\theta).$$

Theorem 3.6. *Assume that the hypotheses* (A1)-(A2) *hold. If*

$$\frac{2q_2^*\varkappa^\alpha}{(1-q_3^*)\Gamma(\alpha+1)}<1,$$

then the implicit fractional problem (1.1)-(1.2) *has at least one solution.*

Proof. This proof is based on the fixed point theorem of Schauder. We establish the proof in several steps.

Step 1: The operator $\aleph : C(\Theta,\mathbb{R})\longrightarrow C(\Theta,\mathbb{R})$ is continuous.
Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence such that $y_n\longrightarrow y$ in $C(\Theta,\mathbb{R})$. Then, for each $\theta\in\Theta$, we have

$$\begin{aligned}|\aleph y_n(\theta)-\aleph y(\theta)| &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa}\rho^{\alpha-1}|\varphi(\rho,y_n(\rho),\varpi_{y_n}(\rho))-\varphi(\rho,y(\rho),\varpi_y(\rho))|d\rho \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^{\varkappa}|\theta-\rho|^{\alpha-1}|\varphi(\rho,y_n(\rho),\varpi_{y_n}(\rho))-\varphi(\rho,y(\rho),\varpi_y(\rho))|d\rho.\end{aligned}$$

By $(A2)$, we have

$$|\varpi_{y_n}(\theta)-\varpi_y(\theta)|\leqslant\psi_1|y_n(\theta)-y(\theta)|+\psi_2|\varpi_{y_n}(\theta)-\varpi_y(\theta)|.$$

Then,

$$|\varpi_{y_n}(\theta) - \varpi_y(\theta)| \leqslant \frac{\psi_1}{1-\psi_2}|y_n(\theta) - y(\theta)|.$$

Thus,

$$\begin{aligned} |\aleph y_n(\theta) - \aleph y(\theta)| \leqslant & \frac{\psi_1}{(1-\psi_2)\Gamma(\alpha)} \int_0^{\varkappa} \rho^{\alpha-1}|y_n(\rho) - y(\rho)|d\rho \\ & + \frac{\psi_1}{(1-\psi_2)\Gamma(\alpha)} \int_0^{\varkappa} |\theta-\rho|^{\alpha-1}|y_n(\rho) - y(\rho)|d\rho. \end{aligned}$$

By applying the Lebesgue dominated convergence theorem, we get

$$|\aleph y_n(\theta) - \aleph y(\theta)| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

which implies that

$$\|\aleph y_n - \aleph y\|_\infty \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

Hence, the operator $\aleph$ is continuous.

Let $R > 0$ and define the ball $D_R = \{y \in C(\Theta, \mathbb{R}) : \|y\|_\infty \leqslant R\}$, where

$$R \geqslant \frac{\psi_3|y_0| + 2\varkappa^\alpha q_1^*}{\psi_3 - 2\varkappa^\alpha q_2^*} \text{ and } \psi_3 := \Gamma(\alpha+1)(1-q_3^*).$$

It is clear that D_R is a bounded, closed and convex subset of $C(\Theta, \mathbb{R})$.

Step 2: $\aleph(D_R) \subset D_R$.
Let $y \in D_R$ and $\theta \in \Theta$, then

$$\begin{aligned} |\aleph y(\theta)| \leqslant |y_0| + & \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} \rho^{\alpha-1}|\varphi(\rho, y(\rho), \varpi_y(\rho))|d\rho \\ & + \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} |\theta-\rho|^{\alpha-1}|\varphi(\rho, y(\rho), \varpi_y(\rho))|d\rho. \end{aligned}$$

From hypothesis $(A2)$, we have

$$\begin{aligned} |\varphi(\theta, y(\theta), \varpi_y(\theta))| &= |\varpi_y(\theta)| \\ &\leqslant q_1(\theta) + q_2^*|y(\theta)| + q_3^*|\varpi_y(\theta)| \\ &\leqslant q_1^* + q_2^* R + q_3^*|\varpi_y(\theta)|. \end{aligned}$$

Then,

$$|\varpi_y(\theta)| \leqslant \frac{q_1^* + q_2^* R}{1-q_3^*}.$$

Finally, we obtain

$$\begin{aligned}
|\aleph y(\theta)| &\leqslant |y_0| + \left[\frac{q_1^* + q_2^* R}{(1-q_3^*)\Gamma(\alpha)}\right] \int_0^{\varkappa} \rho^{\alpha-1} d\rho + \left[\frac{q_1^* + q_2^* R}{(1-q_3^*)\Gamma(\alpha)}\right] \int_0^{\varkappa} |\theta-\rho|^{\alpha-1} d\rho \\
&\leqslant |y_0| + \frac{2\varkappa^{\alpha} q_2^* R}{(1-q_3^*)\Gamma(\alpha+1)} + \frac{2\varkappa^{\alpha} q_1^*}{(1-q_3^*)\Gamma(\alpha+1)} \\
&\leqslant |y_0| + \frac{2\varkappa^{\alpha} q_2^* R}{\psi_3} + \frac{2\varkappa^{\alpha} q_1^*}{\psi_3} \\
&\leqslant R.
\end{aligned}$$

Hence, $\aleph(D_R) \subset D_R$.

Step 3: $\aleph(D_R)$ is equicontinuous.
Let $\theta_1, \theta_2 \in \Theta$ where $\theta_1 < \theta_2$ and $y \in D_R$. Then,

$$\begin{aligned}
&|\aleph y(\theta_2) - \aleph y(\theta_1)| \\
&= \Big| \frac{-1}{\Gamma(\alpha)} \int_0^{\theta_2} \rho^{\alpha-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho - \frac{1}{\Gamma(\alpha)} \int_{\theta_2}^{\varkappa} \rho^{\alpha-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^{\theta_2} (\theta_2-\rho)^{\alpha-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\theta_2}^{\varkappa} (\rho-\theta_2)^{\alpha-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^{\theta_1} \rho^{\alpha-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho + \frac{1}{\Gamma(\alpha)} \int_{\theta_1}^{\varkappa} \rho^{\alpha-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^{\theta_1} (\theta_1-\rho)^{\alpha-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_{\theta_1}^{\varkappa} (\rho-\theta_1)^{\alpha-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho \Big| \\
&\leqslant \frac{2}{\Gamma(\alpha)} \int_{\theta_1}^{\theta_2} \rho^{\alpha-1} |\varphi(\rho, y(\rho), \varpi_y(\rho))| d\rho \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^{\theta_1} [(\theta_2-\rho)^{\alpha-1} - (\theta_1-\rho)^{\alpha-1}] |\varphi(\rho, y(\rho), \varpi_y(\rho))| d\rho \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\theta_1}^{\theta_2} (\theta_2-\rho)^{\alpha-1} |\varphi(\rho, y(\rho), \varpi_y(\rho))| d\rho \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\theta_2}^{\varkappa} [(\rho-\theta_2)^{\alpha-1} - (\rho-\theta_1)^{\alpha-1}] |\varphi(\rho, y(\rho), \varpi_y(\rho))| d\rho \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\theta_1}^{\theta_2} (\rho-\theta_1)^{\alpha-1} |\varphi(\rho, y(\rho), \varpi_y(\rho))| d\rho
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{2(q_1^* + q_2^* R)}{\psi_3}(\theta_2^\alpha - \theta_1^\alpha) + \frac{q_1^* + q_2^* R}{\psi_3}(\theta_2^\alpha - \theta_1^\alpha) \\
&+ \frac{q_1^* + q_2^* R}{\psi_3}(\theta_2 - \theta_1)^\alpha + \frac{q_1^* + q_2^* R}{2\psi_3}[(\varkappa - \theta_2)^\alpha - (\varkappa - \theta_1)^\alpha] \\
&+ \frac{q_1^* + q_2^* R}{\psi_3}(\theta_2 - \theta_1)^\alpha.
\end{aligned}
$$

Then, when $\theta_1 \longrightarrow \theta_2$, the right-hand side of the inequality tend to zero, so we conclude that the operator $\aleph$ is equicontinuous. According to the three steps and the Ascoli-Arzela theorem, we deduce that the operator $\aleph$ has at least a fixed point which is the solution of the problem (1.1)-(1.2). □

4. Examples

Example 4.1. Consider the following implicit fractional problem:

$$
\begin{aligned}
{}^{RC}_{0}D^{\frac{1}{2}}_{1} y(\theta) &= \frac{|y(\theta)| + \left|{}^{RC}_{0}D^{\frac{1}{2}}_{1} y(\theta)\right|}{9\left(1 + |y(\theta)| + \left|{}^{RC}_{0}D^{\frac{1}{2}}_{1} y(\theta)\right|\right)}, \ \theta \in [0,1], \quad (4.1) \\
y(0) &= 1. \quad (4.2)
\end{aligned}
$$

Set

$$
\varphi(\theta, \xi, \eta) = \frac{|\xi| + |\eta|}{9(1 + |\xi| + |\eta|)}, \quad \theta \in [0,1], \ \xi, \eta \in \mathbb{R}.
$$

We observe that φ is a continuous function. And, for any $\xi, \eta, \bar{\xi}, \bar{\eta} \in \mathbb{R}$ and $\theta \in [0,1]$, we have

$$
|\varphi(\theta, \xi, \eta) - \varphi(\theta, \bar{\xi}, \bar{\eta})| \leqslant \frac{1}{9}[|\xi - \bar{\xi}| + |\eta - \bar{\eta}|].
$$

Then, the condition $(A2)$ is satisfied with $\psi_1 = \psi_2 = \frac{1}{9}$. Also, we have

$$
\frac{2\psi_1 \varkappa^\alpha}{(1 - \psi_2)\Gamma(\alpha + 1)} = \frac{2}{(9 - 1)\Gamma(\frac{3}{2})} < 1.
$$

Since all the conditions of Theorem 3.4 are satisfied, then the problem (4.1)-(4.2) has a unique solution on Θ.

Example 4.2. Consider the following implicit fractional problem:

$$
\begin{aligned}
{}^{RC}_{0}D^{\frac{1}{2}}_{1} y(\theta) &= \frac{3\left(\sin\sqrt{|y(\theta)|} + \sin\sqrt{\left|{}^{RC}_{0}D^{\frac{1}{2}}_{1} y(\theta)\right|}\right)}{2(e^{\theta+2} + 15)}, \ \theta \in [0,1], \quad (4.3) \\
y(0) &= 1. \quad (4.4)
\end{aligned}
$$

Set

$$\varphi(\theta,\xi,\eta)=\frac{3\left(\sin\sqrt{|\xi|}+\sin\sqrt{|\eta|}\right)}{2(e^{\theta+2}+15)},\quad \theta\in[0,1],\ \xi,\eta\in\mathbb{R}.$$

It is clear that φ is continuous. For any $\xi,\eta,\bar{\xi},\bar{\eta}\in\mathbb{R}$ and $\theta\in[0,1]$, we have

$$|\varphi(\theta,\xi,\eta)-\varphi(\theta,\bar{\xi},\bar{\eta})|\leqslant\frac{3}{2(e^2+15)}[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|].$$

Hence the assumption $(A2)$ is satisfied with $\psi_1=\psi_2=\frac{3}{2(e^2+15)}$. Also, we have

$$\frac{2\psi_1\varkappa^{\alpha}}{(1-\psi_2)\Gamma(\alpha+1)}=\frac{6}{(2(e^2+15)-3)\Gamma(\frac{3}{2})}<1.$$

Then, the hypotheses of Theorem 3.4 are verified. Consequently the implicit fractional problem (4.3)-(4.4) has a unique solution on Θ.

Example 4.3. Consider the following Cauchy problem:

$$\begin{aligned} {}^{RC}_{0}D^{\frac{1}{2}}_{1}y(\theta) &= \frac{\theta}{25}+\frac{|y(\theta)|+\left|{}^{RC}_{0}D^{\frac{1}{2}}_{1}y(\theta)\right|}{100e^{\theta+6}},\ \theta\in[0,1], && (4.5)\\ y(0) &= 1. && (4.6)\end{aligned}$$

Set

$$\varphi(\theta,\xi,\eta)=\frac{\theta}{25}+\frac{|\xi|+|\eta|}{100e^{\theta+6}},\quad \theta\in[0,1],\ \xi,\eta\in\mathbb{R}.$$

Clearly, φ is a continuous function. For any $\xi,\eta,\bar{\xi},\bar{\eta}\in\mathbb{R}$ and $\theta\in[0,1]$, we have

$$|\varphi(\theta,\xi,\eta)-\varphi(\theta,\bar{\xi},\bar{\eta})|\leqslant\frac{1}{100e^6}[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|].$$

Then, the condition $(A2)$ is satisfied with $\psi_1=\psi_2=\frac{1}{100e^6}$, also we have

$$|\varphi(\theta,\xi,\eta)|\leqslant\frac{1}{25}+\frac{1}{100e^{\theta+6}}|\xi|+\frac{1}{100e^{\theta+6}}|\eta|.$$

So $q_1(\theta)=\frac{1}{25}$, and $q_2^*=q_3^*=\frac{1}{100e^6}<1$.

Moreover,

$$\frac{2q_2^*\varkappa^{\alpha}}{(1-q_3^*)\Gamma(\alpha+1)}=\frac{2}{(100e^6-1)\Gamma(\frac{3}{2})}<1.$$

It follows from Theorem 3.6 that the problem (4.5)-(4.6) has at least one solution on Θ.

5. Implicit Fractional Problem in Banach Space

In this section, we will study a problem similar to problem (1.1)-(1.2) but in a Banach space. Consider the following problem:

$$
\begin{aligned}
{}^{RC}_{0}D^{\gamma}_{\varkappa}y(\theta) &= \varphi(\theta, y(\theta), {}^{RC}_{0}D^{\gamma}_{\varkappa}y(\theta)), \quad \theta \in \Theta := [0, \varkappa], && (5.1)\\
y(0) &= y_0, && (5.2)
\end{aligned}
$$

where ${}^{RC}_{0}D^{\gamma}_{\varkappa}$ is the Riesz-Caputo derivative of order $0 < \gamma \leqslant 1$, $\varphi : \Theta \times E \times E \to E$, $(E, \|.\|)$ is a Banach space and $y_0 \in E$.

Definition 5.1. By a solution of problem (5.1)-(5.2) we mean a function $y \in C(\Theta, E)$ that satisfies the equation (5.1) and the condition (5.2).

Lemma 5.2. *Suppose that the function $\varphi(\theta, \xi, \eta) : \Theta \times E \times E \to E$ is continuous. Then, the problem (5.1)-(5.2) is equivalent to*

$$
\begin{aligned}
y(\theta) = y_0 - \frac{1}{\Gamma(\gamma)} \int_0^{\varkappa} \rho^{\gamma-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho \\
+ \frac{1}{\Gamma(\gamma)} \int_0^{\varkappa} |\theta - \rho|^{\gamma-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho. \qquad (5.3)
\end{aligned}
$$

Let us set the following conditions:

(*A3*) The function $\varphi : \Theta \times E \times E \to E$ is continuous.

(*A4*) There exist constants $\psi_1 > 0$ and $0 < \psi_2 < 1$ such that

$$\|\varphi(\theta, \xi, \eta) - \varphi(\theta, \bar{\xi}, \bar{\eta})\| \leqslant \psi_1 \|\xi - \bar{\xi}\| + \psi_2 \|\eta - \bar{\eta}\|,$$

for any $\xi, \eta, \bar{\xi}, \bar{\eta} \in E$ and $\theta \in \Theta$.

(*A5*) For each $\theta \in \Theta$ and bounded sets $\Omega1, \Omega2 \subseteq E$, we have

$$\alpha(\varphi(\theta, \Omega1, \Omega2)) \leqslant \psi_1 \alpha(\Omega1) + \psi_2 \alpha(\Omega2).$$

Remark 5.3. Let us put

$$q_1(\theta) = \|\varphi(\theta, 0, 0)\|, \ \psi_1 = q_2^*, \ \psi_2 = q_3^*.$$

Then, condition (A4) implies that

$$\|\varphi(\theta, \xi, \eta)\| \leqslant q_1(\theta) + q_2^* \|\xi\| + q_3^* \|\eta\|, \qquad (5.4)$$

for $\theta \in \Theta$, $\xi, \eta \in E$ and $q_1 \in C(\Theta, \mathbb{R}_+)$, with

$$q_1^* =_{\theta \in \Theta} q_1(\theta).$$

Remark 5.4 ([5]). It is worth noting that the hypotheses (A4) and (A5) are equivalent.

Our existence result for the problem (5.1)-(5.2) is based on the concept of measures of noncompactness and Mönch's fixed point theorem.

Theorem 5.5. *Assume* (A3)-(A4) *are verified. If*

$$\frac{2q_2^*\varkappa^\gamma}{(1-q_3^*)\Gamma(\gamma+1)} < 1,$$

then the problem (5.1)-(5.2) *has at least one solution.*

Proof. Transform problem (5.1)-(5.2) into a fixed point problem by considering the operator $\aleph : C(\Theta, E) \longrightarrow C(\Theta, E)$ by

$$\begin{aligned}\aleph y(\theta) = y_0 &- \frac{1}{\Gamma(\gamma)}\int_0^{\varkappa} \rho^{\gamma-1}\varphi(\rho, y(\rho), \varpi_y(\rho))d\rho \\ &+ \frac{1}{\Gamma(\gamma)}\int_0^{\varkappa} |\theta-\rho|^{\gamma-1}\varphi(\rho, y(\rho), \varpi_y(\rho))d\rho.\end{aligned}$$

The proof will be given in several steps.

Step 1: $\varkappa$ is continuous.
Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence such that $y_n \longrightarrow y$ in $\in C(\Theta, E)$, then for each $\theta \in \Theta$, we have

$$\begin{aligned}\|\aleph y_n(\theta) - \aleph y(\theta)\| \leqslant &\frac{1}{\Gamma(\gamma)}\int_0^{\varkappa} \rho^{\gamma-1}\|\varphi(\rho, y_n(\rho), \varpi_{y_n}(\rho)) - \varphi(\rho, y(\rho), \varpi_y(\rho))\|d\rho \\ &+ \frac{1}{\Gamma(\gamma)}\int_0^{\varkappa} |\theta-\rho|^{\gamma-1}\|\varphi(\rho, y_n(\rho), \varpi_{y_n}(\rho)) - \varphi(\rho, y(\rho), \varpi_y(\rho))\|d\rho.\end{aligned}$$

By (A4), we have

$$\|\varpi_{y_n}(\theta) - \varpi_y(\theta)\| \leqslant \psi_1\|y_n(\theta) - y(\theta)\| + \psi_2\|\varpi_{y_n}(\theta) - \varpi_y(\theta)\|.$$

Then,

$$\|\varpi_{y_n}(\theta) - \varpi_y(\theta)\| \leqslant \frac{\psi_1}{1-\psi_2}\|y_n(\theta) - y(\theta)\|.$$

Thus, we obtain

$$\begin{aligned}\|\aleph y_n(\theta) - \aleph y(\theta)\| \leqslant &\frac{\psi_1}{(1-\psi_2)\Gamma(\gamma)}\int_0^{\varkappa} \rho^{\gamma-1}\|y_n(\rho) - y(\rho)\|d\rho \\ &+ \frac{\psi_1}{(1-\psi_2)\Gamma(\gamma)}\int_0^{\varkappa} |\theta-\rho|^{\gamma-1}\|y_n(\rho) - y(\rho)\|d\rho.\end{aligned}$$

By applying the Lebesgue dominated convergence theorem, we obtain

$$\|\aleph y_n(\theta) - \aleph y(\theta)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

which implies that

$$\|\aleph y_n - \aleph y\|_\infty \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Consequently, the operator $\aleph$ is continuous.

Let $R > 0$ and define the ball $D_R = \{y \in C(\Theta, E) : \|y\|_\infty \leqslant R\}$, where

$$R \geqslant \frac{\overline{\psi_3}\|y_0\| + 2\varkappa^\gamma q_1^*}{\overline{\psi_3} - 2\varkappa^\gamma q_2^*} \text{ and } \overline{\psi_3} := \Gamma(\gamma+1)(1-q_3^*).$$

Obviously D_R is a bounded, closed and convex subset of $C(\Theta, E)$.

Step 2: $\aleph(D_R) \subset D_R$.
Let $y \in D_R$ and $\theta \in \Theta$, then

$$\begin{aligned}\|\aleph y(\theta)\| \leqslant \|y_0\| &+ \frac{1}{\Gamma(\gamma)}\int_0^\varkappa \rho^{\gamma-1}\|\varphi(\rho, y(\rho), \varpi_y(\rho))\|d\rho \\ &+ \frac{1}{\Gamma(\gamma)}\int_0^\varkappa |\theta-\rho|^{\gamma-1}\|\varphi(\rho, y(\rho), \varpi_y(\rho))\|d\rho.\end{aligned}$$

From hypothesis $(A4)$, we have

$$\begin{aligned}\|\varphi(\theta, y(\theta), \varpi_y(\theta))\| &= \|\varpi_y(\theta)\| \\ &\leqslant q_1(\theta) + q_2^*\|y(\theta)\| + q_3^*\|\varpi_y(\theta)\| \\ &\leqslant q_1^* + q_2^* R + q_3^*\|\varpi_y(\theta)\|.\end{aligned}$$

Then,

$$\|\varpi_y(\theta)\| \leqslant \frac{q_1^* + q_2^* R}{1-q_3^*}.$$

Finally, we have

$$\begin{aligned}\|\aleph y(\theta)\| &\leqslant \|y_0\| + \frac{2\varkappa^\gamma q_1^*}{\Gamma(\gamma+1)(1-q_3^*)} + \frac{2\varkappa^\gamma q_2^* R}{\Gamma(\gamma+1)(1-q_3^*)} \\ &\leqslant \|y_0\| + \frac{2\varkappa^\gamma q_1^*}{\overline{\psi_3}} + \frac{2\varkappa^\gamma q_2^* R}{\overline{\psi_3}} \\ &\leqslant R.\end{aligned}$$

As a consequence, $\aleph(D_R) \subset D_R$.

Step 3: $\aleph(D_R)$ is equicontinuous.
Let $\theta_1, \theta_2 \in \Theta$ such that $\theta_1 < \theta_2$ and $y \in D_R$. Then

$$\begin{aligned}&\|\aleph y(\theta_2) - \aleph y(\theta_1)\| \\ &\leqslant \frac{1}{\Gamma(\gamma)}\int_{\theta_1}^{\theta_2} \rho^{\gamma-1}\|\varphi(\rho, y(\rho), \varpi_y(\rho))\|d\rho \\ &\quad + \frac{1}{\Gamma(\gamma)}\int_0^{\theta_1} [(\theta_2-\rho)^{\gamma-1} - (\theta_1-\rho)^{\gamma-1}]\|\varphi(\rho, y(\rho), \varpi_y(\rho))\|d\rho \\ &\quad + \frac{1}{\Gamma(\gamma)}\int_{\theta_1}^{\theta_2} (\theta_2-\rho)^{\gamma-1}\|\varphi(\rho, y(\rho), \varpi_y(\rho))\|d\rho \\ &\quad + \frac{1}{\Gamma(\gamma)}\int_{\theta_2}^{\varkappa} [(\rho-\theta_2)^{\gamma-1} - (\rho-\theta_1)^{\gamma-1}]\|\varphi(\rho, y(\rho), \varpi_y(\rho))\|d\rho\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\gamma)}\int_{\theta_1}^{\theta_2}(\rho-\theta_1)^{\gamma-1}\|\varphi(\rho,y(\rho),\varpi_y(\rho))\|d\rho\\
&\leqslant \frac{2(q_1^*+q_2^*R)}{\overline{\psi_3}}(\theta_2^\gamma-\theta_1^\gamma)+\frac{q_1^*+q_2^*R}{\overline{\psi_3}}(\theta_2^\gamma-\theta_1^\gamma)\\
&+\frac{q_1^*+q_2^*R}{\overline{\psi_3}}(\theta_2-\theta_1)^\gamma+\frac{q_1^*+q_2^*R}{\overline{\psi_3}}[(\varkappa-\theta_2)^\gamma-(\varkappa-\theta_1)^\gamma]\\
&+\frac{q_1^*+q_2^*R}{\overline{\psi_3}}(\theta_2-\theta_1)^\gamma.
\end{aligned}
$$

As $\theta_1 \longrightarrow \theta_2$, the right-hand side of the preceding inequality tend to zero, then $\aleph(D_R)$ is equicontinuous.

Step 4: The implication of Mönch's theorem is satisfied.
Let Ω be a subset of $\aleph(D_R)$ and $b(\theta)=\zeta(\Omega(\theta))$ a continuous function on Θ.
For $\theta\in\Theta$, and by Lemma 2.7, the function b can be given by

$$
\begin{aligned}
b(\theta) &= \zeta(\Omega(\theta))\\
&= \zeta\{\aleph y(\theta),\ y\in\Omega\}\\
&= \zeta\left\{y_0-\frac{1}{\Gamma(\gamma)}\int_0^\varkappa \rho^{\gamma-1}\varphi(\rho,y(\rho),\varpi_y(\rho))d\rho\right.\\
&\quad\left.+\frac{1}{\Gamma(\gamma)}\int_0^\varkappa |\theta-\rho|^{\gamma-1}\varphi(\rho,y(\rho),\varpi_y(\rho))d\rho,\ \ y\in\Omega\right\}\\
&\leqslant \frac{1}{\Gamma(\gamma)}\zeta\left\{\int_0^\varkappa \rho^{\gamma-1}\varphi(\rho,y(\rho),\varpi_y(\rho))d\rho,\ \ y\in\Omega\right\}\\
&\quad+\frac{1}{\Gamma(\gamma)}\zeta\left\{\int_0^\varkappa |\theta-\rho|^{\gamma-1}\varphi(\rho,y(\rho),\varpi_y(\rho))d\rho,\ \ y\in\Omega\right\}\\
&\leqslant \frac{1}{\Gamma(\gamma)}\int_0^\varkappa \rho^{\gamma-1}\left\{\alpha\Big(\varphi(\rho,y(\rho),\varpi_y(\rho))\Big)d\rho,\ \ y\in\Omega\right\}\\
&\quad+\frac{1}{\Gamma(\gamma)}\int_0^\varkappa |\theta-\rho|^{\gamma-1}\left\{\zeta\Big(\varphi(\rho,y(\rho),\varpi_y(\rho))\Big)d\rho,\ \ y\in\Omega\right\}.
\end{aligned}
$$

By condition $(A5)$, we obtain

$$
\begin{aligned}
\zeta(\varphi(\theta,y(\theta),\varpi_y(\theta))) &= \zeta(\varpi_y(\theta))\\
&\leqslant \psi_1\zeta(y(\theta))+\psi_2\zeta(\varpi_y(\theta)).
\end{aligned}
$$

Thus,

$$
\zeta(\varpi_y(\theta))\leqslant \frac{k}{1-\psi_2}\zeta(y(\theta)).
$$

Then,

$$\begin{aligned} b(\theta) &= \zeta(\Omega(\theta)) \\ &\leqslant \frac{\psi_1}{(1-\psi_2)\Gamma(\gamma)} \int_0^{\varkappa} \rho^{\gamma-1}\{\zeta(y(\rho))d\rho, y \in \Omega\} \\ &\quad + \frac{\psi_1}{(1-\psi_2)\Gamma(\gamma)} \int_0^{\varkappa} |\theta-\rho|^{\gamma-1}\{\zeta(y(\rho))d\rho, y \in \Omega\} \\ &\leqslant \frac{2\varkappa^{\gamma}\psi_1}{(1-\psi_2)\Gamma(\gamma+1)}\zeta_c(\Omega). \end{aligned}$$

Therefore,

$$\zeta_c(\Omega) \leqslant \frac{2\varkappa^{\gamma}\psi_1}{(1-\psi_2)\Gamma(\gamma+1)}\zeta_c(\Omega),$$

and by Remark 5.3, we have

$$\zeta_c(\Omega) \leqslant \frac{2q_2^*\varkappa^{\gamma}}{(1-q_3^*)\Gamma(\gamma+1)}\zeta_c(\Omega),$$

which implies that $\zeta_c(\Omega) = 0$. We conclude then, that $\aleph$ has a fixed point that is the solution of the problem (5.1)-(5.2), according to Mönch's fixed point theorem. □

5.1. Ulam-Hyers Stability

In this section, we will establish the Ulam stability for the problem (5.1)-(5.2).

Definition 5.6 ([23])**.** The problem (5.1)-(5.2) is Ulam-Hyers stable if there exists a real number $C_{\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C(\Theta, E)$ of the inequality

$$\left\| {}_0^{RC}D_{\varkappa}^{\gamma}y(\theta) - \varphi(\theta, y(\theta), {}_0^{RC}D_{\varkappa}^{\gamma}y(\theta)) \right\| < \varepsilon, \quad \theta \in \Theta, \tag{5.5}$$

there exists a solution $\bar{y} \in C(\Theta, E)$ of the problem (5.1)-(5.2) with

$$\|y(\theta) - \bar{y}(\theta))\| < C_{\varphi}\varepsilon, \quad \theta \in \Theta.$$

Definition 5.7 ([23])**.** The problem (5.1)-(5.2) is generalized Ulam-Hyers stable if there exists $\phi_{\varphi} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\phi_{\varphi}(0) = 0$ such that for each solution $y \in C(\Theta, E)$ of the inequality (5.5) there exists a solution $\bar{y} \in C(\Theta, E)$ of the problem (5.1)-(5.2) with

$$\|y(\theta) - \bar{y}(\theta))\| < \phi_{\varphi}\varepsilon, \quad \theta \in \Theta.$$

Remark 5.8 ([23]). A function $y \in C(\Theta, E)$ is a solution of the inequality (5.5) if and only if there exists a function $\ell \in C(\Theta, E)$ (which depend on y) such that

1. $\|\ell(\theta)\| \leqslant \varepsilon, \quad \theta \in \Theta.$
2. ${}_0^{RC}D_{\varkappa}^{\gamma}y(\theta) = \varphi(\theta, \xi(\theta), {}_0^{RC}D_{\varkappa}^{\gamma}y(\theta)) + \ell(\theta), \quad \theta \in \Theta.$

Lemma 5.9. *The solution of the following perturbed problem*

$$\begin{aligned} {}_0^{RC}D_{\varkappa}^{\gamma} y(\theta) &= \varphi(\theta, y(\theta), {}_0^{RC}D_{\varkappa}^{\gamma} y(\theta)) + \ell(\theta), \quad \theta \in \Theta := [0, \varkappa], \\ y(0) &= y_0, \end{aligned}$$

is given by

$$y(\theta) = y_0 - \frac{1}{\Gamma(\gamma)} \int_0^{\varkappa} \rho^{\gamma-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho + \frac{1}{\Gamma(\gamma)} \int_0^{\varkappa} |\theta - \rho|^{\gamma-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho$$
$$- \frac{1}{\Gamma(\gamma)} \int_0^{\varkappa} \rho^{\gamma-1} \ell(\rho) d\rho + \frac{1}{\Gamma(\gamma)} \int_0^{\varkappa} |\theta - \rho|^{\gamma-1} \ell(\rho) d\rho.$$

Moreover, the solution satisfies the following inequality

$$\left\| y(\theta) - \left[y_0 - \frac{1}{\Gamma(\gamma)} \int_0^{\varkappa} \rho^{\gamma-1} \varpi_y(\rho) d\rho + \frac{1}{\Gamma(\gamma)} \int_0^{\varkappa} |\theta - \rho|^{\gamma-1} \varpi_y(\rho) d\rho \right] \right\|$$
$$\leqslant \Psi \varepsilon,$$

where $\theta \in \Theta$ *and* $\Psi = \frac{2\varkappa^{\gamma}}{\Gamma(\gamma+1)}$.

Theorem 5.10. *Assume that* (A3)-(A4) *and*

$$\frac{2\psi_1 \varkappa^{\gamma}}{(1-\psi_2)\Gamma(\gamma+1)} < 1,$$

hold. Then the problem (5.1)-(5.2) *is Ulam-Hyers stable.*

Proof. Let $y \in C(\Theta, E)$ be a solution of the inequality (5.5) and $\bar{y} \in C(\Theta, E)$ the unique solution of the problem (5.1)-(5.2), then

$$\begin{aligned} \|y(\theta) - \bar{y}(\theta)\| &= \left\| y(\theta) - \left[y_0 - \frac{1}{\Gamma(\gamma)} \int_0^{\varkappa} \rho^{\gamma-1} \varphi(\rho, \bar{y}(\rho), \varpi_{\bar{y}}(\rho)) d\rho \right.\right. \\ &+ \left.\left. \frac{1}{\Gamma(\gamma)} \int_0^{\varkappa} |\theta - \rho|^{\gamma-1} \varphi(\rho, \bar{y}(\rho), \varpi_{\bar{y}}(\rho)) d\rho \right] \right\| \\ &\leqslant \left\| y(\theta) - \left[y_0 - \frac{1}{\Gamma(\gamma)} \int_0^{\varkappa} \rho^{\gamma-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho \right.\right. \\ &+ \left.\left. \frac{1}{\Gamma(\gamma)} \int_0^{\varkappa} |\theta - \rho|^{\gamma-1} \varphi(\rho, y(\rho), \varpi_y(\rho)) d\rho \right] \right\| \\ &+ \frac{1}{\Gamma(\gamma)} \left\| \int_0^{\varkappa} \rho^{\gamma-1} (\varphi(\rho, y(\rho), \varpi_y(\rho)) - \varphi(\rho, \bar{y}(\rho), \varpi_{\bar{y}}(\rho))) d\rho \right\| \\ &+ \frac{1}{\Gamma(\gamma)} \left\| \int_0^{\varkappa} |\theta - \rho|^{\gamma-1} (\varphi(\rho, y(\rho), \varpi_y(\rho)) - \varphi(\rho, \bar{y}(\rho), \varpi_{\bar{y}}(\rho))) d\rho \right\|. \end{aligned}$$

By hypothesis (A4), we have

$$\|\varphi(\rho, y(\rho), \varpi_y(\rho)) - \varphi(\rho, \bar{y}(\rho), \varpi_{\bar{y}}(\rho))\| \leqslant \psi_1 \|y(\theta) - \bar{y}(\theta)\| + \psi_2 \|\varpi_y(\theta) - \varpi_{\bar{y}}(\theta)\|.$$

Then,

$$\|\varpi_y(\theta) - \varpi_{\bar{y}}(\theta)\| \leqslant \frac{\psi_1}{1-\psi_2} \|y(\theta) - \bar{y}(\theta)\|.$$

Thus,

$$\begin{aligned}\|y(\theta)-\bar{y}(\theta)\| &\leqslant \frac{2\varkappa^{\gamma}\varepsilon}{\Gamma(\gamma+1)}+\frac{2\varkappa^{\gamma}\psi_1}{(1-\psi_2)\Gamma(\gamma+1)}\|y-\bar{y}\| \\ &\leqslant \Psi\varepsilon+\frac{\psi_1\Psi}{1-\psi_2}\|y-\bar{y}\|,\end{aligned}$$

which implies that

$$\|y-\bar{y}\| \leqslant \frac{\Psi\varepsilon}{1-\frac{\psi_1\Psi}{1-\psi_2}} := C_{\varphi}\,\varepsilon.$$

Consequently, the problem (5.1)-(5.2) is Ulam-Hyers stable.
If we take $\phi_{\varphi}(\varepsilon)=C_{\varphi}\varepsilon$, $\phi_{\varphi}(0)=0$ then we get the generalized Ulam-Hyers stability of the problem (5.1)-(5.2). □

6. Examples

Set

$$E=l^1=\left\{y=(y_1,y_2,\ldots,y_n,\ldots),\ \sum_{n=1}^{\infty}|y_n|<\infty\right\},$$

where E is a Banach space with the norme $\|y\|=\sum_{n=1}^{\infty}|y_n|$.

Example 6.1. Consider the following Cauchy problem:

$$\begin{aligned}{}^{RC}_{0}D^{\frac{1}{2}}_{1}y_n(\theta) &= \frac{7+|y_n(\theta)|+\frac{1}{2}\left|{}^{RC}_{0}D^{\frac{1}{2}}_{1}y_n(\theta)\right|}{48e^{\theta+1}\left(1+\|y(\theta)\|+\left\|{}^{RC}_{0}D^{\frac{1}{2}}_{1}y(\theta)\right\|\right)}, \quad \text{for each } \theta\in[0,1], \quad (6.1)\\ y_n(0) &= 1. \quad (6.2)\end{aligned}$$

Set

$$\varphi(\theta,\xi,\eta)=\frac{7+\|\xi\|+\frac{1}{2}\|\eta\|}{48e^{\theta+1}(1+\|\xi\|+\|\eta\|)}, \quad \theta\in[0,1],\ \xi,\eta\in E.$$

Clearly, φ is a continuous function. And, for any $\xi,\eta,\bar{\xi},\bar{\eta}\in E$ and $\theta\in[0,1]$, we have

$$\|\varphi(\theta,\xi,\eta)-\varphi(\theta,\bar{\xi},\bar{\eta})\| \leqslant \frac{1}{48e}\|\xi-\bar{\xi}\|+\frac{1}{96e}\|\eta-\bar{\eta}\|.$$

Then, the Assumption $(A4)$ is satisfied by $\psi_1=\frac{1}{48e}$ and $\psi_2=\frac{1}{96e}$. Also

$$\|\varphi(\theta,\xi,\eta)\| \leqslant \frac{1}{48e^{\theta+1}}(7+\|\xi\|+\frac{1}{2}\|\eta\|).$$

Thus, the condition (5.4) is satisfied with $q_1(\theta)=\frac{7}{48e^{\theta+1}}$ and $q_2^*=q_3^*=\frac{1}{96e}<1$. Moreover,

$$\frac{2q_2^*\varkappa^{\gamma}}{(1-q_3^*)\Gamma(\gamma+1)}=\frac{\frac{2}{48e}}{(1-\frac{1}{96e})\Gamma(\frac{3}{2})}<1.$$

Since the conditions of Theorem 5.5 are satisfied, the problem (6.1)-(6.2) has at least one solution. And, as

$$\frac{\Psi\psi_1}{1-\psi_2} = \frac{2\psi_1\varkappa^\gamma}{(1-\psi_2)\Gamma(\gamma+1)} = \frac{\frac{2}{48e}}{(1-\frac{1}{96e})\Gamma(\frac{3}{2})} < 1,$$

then by Theorem 5.10, we can deduce that our problem is Ulam-Hyers stable.

Example 6.2. Consider the following Cauchy problem:

$$\begin{aligned} {}^{RC}_{0}D^{\frac{1}{2}}_{1}y_n(\theta) &= \frac{2\cos(\theta)+|y_n(\theta)|+\left|{}^{RC}_{0}D^{\frac{1}{2}}_{1}y_n(\theta)\right|}{183e^{\sqrt{\theta+1}}\left(1+\|y(\theta)\|+\left\|{}^{RC}_{0}D^{\frac{1}{2}}_{1}y(\theta)\right\|\right)}, \quad \text{for each } \theta\in[0,1], \quad (6.3)\\ y_n(0) &= 1. \quad (6.4)\end{aligned}$$

Set

$$\varphi(\theta,\xi,\eta) = \frac{2\cos(\theta)+\|\xi\|+\|\eta\|}{183e^{\sqrt{\theta+1}}(1+\|\xi\|+\|\eta\|)}, \quad \theta\in[0,1], \ \xi,\eta\in E.$$

Clearly, φ is a continuous function. For any $\xi,\eta,\bar{\xi},\bar{\eta}\in E$ and $\theta\in[0,1]$, we have

$$\|\varphi(\theta,\xi,\eta)-\varphi(\theta,\bar{\xi},\bar{\eta})\| \leqslant \frac{1}{183e}[\|\xi-\bar{\xi}\|+\|\eta-\bar{\eta}\|].$$

Then, the hypothesis $(A4)$ is satisfied by $\psi_1=\psi_2=\frac{1}{183e}$. Also we have

$$\|\varphi(\theta,\xi,\eta)\| \leqslant \frac{1}{183e^{\sqrt{\theta+1}}}(2\cos(\theta)+\|\xi\|+\|\eta\|).$$

So $q_1(\theta)=\frac{2\cos(\theta)}{183e^{\sqrt{\theta+1}}}$ and $q_2^*=q_3^*=\frac{1}{183e}<1.$

And as

$$\frac{2q_2^*\varkappa^\gamma}{(1-q_3^*)\Gamma(\gamma+1)} = \frac{\frac{2}{183e}}{(1-\frac{1}{183e})\Gamma(\frac{3}{2})} < 1.$$

Thus, by Theorem 5.5, the problem (6.3)-(6.4) has at least one solution.

Moreover,

$$\frac{\Psi\psi_1}{1-\psi_2} = \frac{2\psi_1\varkappa^\gamma}{(1-\psi_2)\Gamma(\gamma+1)} = \frac{\frac{2}{183e}}{(1-\frac{1}{183e})\Gamma(\frac{3}{2})} < 1.$$

Then, Theorem 5.10 assures that our problem is Ulam-Hyers stable.

7. Conclusion

In this paper, we have made a substantial contribution to the study of certain classes of fractional differential equations involving the Riesz-Caputo fractional derivative. The methodologies utilized are primarily grounded in fixed point theorems, such as those of Schauder and Banach, as well as the technique of measure of noncompactness. We have investigated Ulam's stability of these problems, advancing the understanding of fractional differential equations under various conditions. In future research, we aim to explore additional classes of fractional differential equations and inclusions, including problems with retarded (delayed) and advanced arguments, as well as impulsive problems, focusing on both instantaneous and non-instantaneous impulses.

Declarations

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Competing interests It is declared that authors has no competing interests.

Author's contributions The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

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