Journal of Fractional Calculus and Nonlinear Systems
jfcns.sabapub.com
ISSN : 2709-9547

J Frac Calc & Nonlinear Sys
(2024)5(2) : 53-68
doi:10.48185/jfcns.v5i2.1269

A new generalized class of fractional operators with weight and respect to another function

Khalid Hattaf [a,b]

[a] Equipe de Recherche en Modélisation et Enseignement des Mathématiques (ERMEM), Centre Régional des Métiers de l'Education et de la Formation (CRMEF), 20340 Derb Ghalef, Casablanca, Morocco

[b] Laboratory of Analysis, Modeling and Simulation (LAMS), Faculty of Sciences Ben M'Scik, Hassan II University of Casablanca, P.O Box 7955 Sidi Othman, Casablanca, Morocco

• *Received: 26 July 2024* • *Accepted: 09 December 2024* • *Published Online: 30 December 2024*

Abstract

This paper introduces and investigates the properties of a new generalized class of fractional differential and integral operators. Such newly class covers various definitions of fractional derivatives with singular and non-singular kernels, weighted fractional derivatives with respect to another function, as well as the new mixed fractional derivative in the sense of Caputo and Riemann-Liouville. Furthermore, the newly introduced class includes all existing forms of fractional integrals, weighted fractional integrals and also weighted fractional integrals with respect to another function, including Riemann-Liouville, Hadamard, Katugampola, and Hattaf fractional integrals. Moreover, some fundamental properties of the new generalized class of fractional differential and integral operators are rigorously derived.

Keywords: Fractional calculus, singular and non-singular kernels, weighted Laplace transform, weighted fractional operator with respect to another function.
2010 MSC: 26A33, 34A08, 44A10.

1. Introduction

Fractional calculus is one of the fastest growing areas of research today, which is a branch of mathematical analysis that studies the integration and differentiation of non-integer orders. Such branch of mathematics has many applications in various fields such as physics, engineering, signal processing, control systems and biology. In addition, fractional calculus provides a powerful tool to model systems with memory effects or systems that exhibit nonlocal behavior, offering a more accurate representation of many real-world phenomena. For example, Shah and Abdeljawad [1] proposed a fractal-fractional model for emissions of carbon dioxide (CO_2) from energy sector. In [2], the authors investigated

*Corresponding author: k.hattaf@yahoo.fr

the dynamical behavior of rotavirus infectious disease model via piecewise modified ABC fractional order derivative. Other recent applications of fractional derivatives have been used in [3, 4, 5, 6].

Nowadays, there exists a large number of definitions for nonlocal fractional derivatives that are used to model the nonlocal behaviors of many dynamical systems arising from different fields of science and engineering. Mainly, there are two classes of such types of nonlocal fractional derivatives. The first one has a singular kernel like the Caputo fractional derivative introduced by Caputo in 1967 [7], the Riemann-Liouville fractional derivative [8, 9], the Hadamard fractional derivative [10, 11] and the Katugampola fractional derivative [12]. However, the second class has a non-singular kernel like the Caputo-Fabrizio (CF) fractional derivative [13], the Atangana-Baleanu (AB) fractional derivative [14], the weighted AB fractional derivative [15], the generalized Hattaf fractional (GHF) derivative [16] and the weighted CF fractional derivative with respect to another function [17].

On the other hand, there are several forms of fractional integrals available in the literature. The most popular of them are the Riemann-Liouville fractional integral [8], the Hadamard fractional integral [18, 19] and the Katugampola fractional integral [20]. There are also other recently kinds of fractional integrals such as the GHF integral [16], the fractional integral corresponding to the AB fractional derivative with the generalized Mittag-Leffler function [21], the weighted AB fractional integral [15], the AB fractional integral with respect to another function [22], the AB fractional integral [14], the weighted CF fractional integral with respect to another function [17], the CF fractional integral [13], the fractional integral corresponding to the new mixed fractional derivative [23], the fractional integrals introduced in [24, 25], the modified fractional integral [26], the weighted Riemann-Liouville fractional integral with respect to another [27], the Riemann-Liouville fractional integral with respect to another function [28, 29, 8], as well as the tempered fractional integral [30, 31].

The current study aimed to introduce a new class of fractional differential and integral operators in order to generalize all definitions of fractional and integrals cited above. Additionally, the new introduced class aims to cover the new recent Hattaf mixed fractional derivative introduced in [23] and also the weighted fractional derivatives and integrals with respect to another function presented in [22, 32].

The rest of present paper is structured as follows. Section 2 introduces the new generalized class of fractional derivatives in the sense of Caputo and Riemann-Liouville, as well as its special cases. Section 3 presents the fractional integral associated to the newly introduced class of fractional derivatives and its particular cases. Section 4 focuses on the fondamental properties of the newly generalized class of fractional differential and integral operators. Finally, the conclusion is presented in Section 5.

2. The new class of fractional derivatives

This section focuses on the definition of the new generalized class of fractional derivatives in the sense of Caputo and Riemann-Liouville.

Definition 2.1. Let $(p,q) \in [0,1]^2$, $r, m > 0$, $Re(\mu) > 0$, $\sigma \in \mathbb{R}$, $\delta \in \mathbb{R}^*$ and $f \in \mathcal{H}^1(a,b)$. The generalized mixed fractional derivative of the function $f(t)$ of order p in Caputo sense with the weight function $w(t)$ and respect to another function $\phi(t)$ is given by

$$ {}^{C}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t) = \frac{H(p+q-1)}{(2-p-q)w(t)} \int_a^t \big(\phi(t)-\phi(\tau)\big)^{\mu-1} E^{\sigma}_{r,\mu}[-\lambda^{\delta}_{p,q}\big(\phi(t)-\phi(\tau)\big)^{m}](wf)'(\tau)d\tau, \tag{2.1} $$

where $w, \phi \in C^1(a,b)$, $w, \phi' > 0$ on [a,b], $H(.)$ is a normalization function such that $H(0) = H(1) = 1$, $\lambda^{\delta}_{p,q} = \dfrac{\delta(p+q-1)}{2-p-q}$ and $E^{\sigma}_{r,\mu}(t) = \displaystyle\sum_{k=0}^{+\infty} \frac{(\sigma)_k t^k}{k!\Gamma(rk+\mu)}$ is the generalized Mittag-Leffler function of three parameters [33] with $(\sigma)_0 = 1$ and $(\sigma)_k = \sigma(\sigma+1)\cdots(\sigma+k-1)$ is the Pochhammer symbol.

Remark 2.2. Definition 2.1 covers a great number of definitions of fractional derivatives with singular and non-singular kernels. For example,

1. When $q = \delta = 1$ in Eq. (2.1), we have the new weighted fractional derivative with respect to another function [32] given by
$$ {}^{C}D^{p,1,r,m,\mu}_{a,\sigma,1,w,\phi}f(t) = \frac{H(p)}{(1-p)w(t)} \int_a^t \big(\phi(t)-\phi(\tau)\big)^{\mu-1} E^{\sigma}_{r,\mu}[-\lambda^{1}_{p,1}\big(\phi(t)-\phi(\tau)\big)^{m}](wf)'(\tau)d\tau, $$
where $\lambda^1_{p,1} = \frac{p}{1-p}$.
2. When $q = \delta = \mu = \sigma = 1$ and $\phi(t) = t$ in Eq. (2.1), we have the GHF derivative [16] given by
$$ {}^{C}D^{p,1,r,m,1}_{a,1,1,w,t}f(t) = \frac{H(p)}{(1-p)w(t)} \int_a^t E^{1}_{r,1}[-\lambda^{1}_{p,1}(t-\tau)^{m}](wf)'(\tau)d\tau. $$
Notice that $E^1_{r,1}(t) = E_r(t)$.
3. When $r = m = p$, $q = \delta = 1$, $w(t) = 1$ and $\phi(t) = t$ in Eq. (2.1), we have the generalized AB fractional derivative with generalized Mittag-Leffler function [21] given by
$$ {}^{C}D^{p,1,p,p,\mu}_{a,\sigma,1,1,t}f(t) = \frac{H(p)}{1-p} \int_a^t (t-\tau)^{\mu-1} E^{\sigma}_{p,\mu}[-\lambda^{1}_{p,1}(t-\tau)^{p}]f'(\tau)d\tau. $$
4. When $r = m = p$, $q = \delta = \mu = \sigma = 1$ and $\phi(t) = t$, we have the weighted AB fractional derivative [15] given by
$$ {}^{C}D^{p,1,p,p,1}_{a,1,1,w,t}f(t) = \frac{H(p)}{(1-p)w(t)} \int_a^t E^{1}_{r,1}[-\lambda^{1}_{p,1}(t-\tau)^{p}](wf)'(\tau)d\tau. $$
5. When $r = m = p$, $q = \delta = \mu = \sigma = 1$ and $w(t) = 1$, we have the AB fractional derivative with respect to another function [22] given by
$$ {}^{C}D^{p,1,p,p,1}_{a,1,1,1,\phi}f(t) = \frac{H(p)}{1-p} \int_a^t E^{1}_{p,1}[-\lambda^{1}_{p,1}\big(\phi(t)-\phi(\tau)\big)^{p}]f'(\tau)d\tau. $$
6. When $r = m = p$, $q = \delta = \mu = \sigma = 1$, $w(t) = 1$ and $\phi(t) = t$, we have the AB fractional derivative [14] given by
$$ {}^{C}D^{p,1,p,p,1}_{a,1,1,1,t}f(t) = \frac{H(p)}{1-p} \int_a^t E^{1}_{p,1}[-\lambda^{1}_{p,1}(t-\tau)^{p}]f'(\tau)d\tau. $$

7. When $r = m = q = \delta = \mu = \sigma = 1$, we have the weighted CF fractional derivative with respect to another function [17] given by

$$^{C}D^{p,1,1,1,1}_{a,1,1,w,\phi}f(t) = \frac{H(p)}{(1-p)w(t)}\int_a^t E^1_{1,1}[-\lambda^1_{p,1}\big(\phi(t)-\phi(\tau)\big)](wf)'(\tau)d\tau.$$

Notice that $E^1_{1,1}(t) = \exp(t)$.

8. When $r = m = q = \delta = \mu = \sigma = 1$ and $w(t) = 1$, we have the CF fractional derivative with respect to another function [17] given by

$$^{C}D^{p,1,1,1,1}_{a,1,1,1,\phi}f(t) = \frac{H(p)}{1-p}\int_a^t E^1_{1,1}[-\lambda^1_{p,1}\big(\phi(t)-\phi(\tau)\big)]f'(\tau)d\tau.$$

9. When $r = m = q = \delta = \mu = \sigma = 1$, $w(t) = 1$ and $\phi(t) = t$, we have the CF fractional derivative [13] given by

$$^{C}D^{p,1,1,1,1}_{a,1,1,1,t}f(t) = \frac{H(p)}{1-p}\int_a^t E^1_{1,1}[-\lambda^1_{p,1}(t-\tau)]f'(\tau)d\tau.$$

10. When $\mu = q$, $\sigma = 1$ and $\phi(t) = t$, we have the Hattaf mixed fractional derivative [23] given by

$$^{C}D^{p,q,r,m,q}_{a,1,\delta,w,t}f(t) = \frac{H(p+q-1)}{(2-p-q)w(t)}\int_a^t (t-\tau)^{q-1}E^1_{r,q}[-\lambda^\delta_{p,q}(t-\tau)^m](wf)'(\tau)d\tau.$$

Notice that $E^1_{r,q}(t) = E_{r,q}(t)$.

11. When $\mu = q = \sigma = 1$, $m = r$, $\delta = \ln(\bar{p})$ (with $\bar{p} > 0$) and $\phi(t) = t$, we have the power fractional derivative [24] given by

$$^{C}D^{p,1,r,r,1}_{a,1,\ln(\bar{p}),w,t}f(t) = \frac{H(p)}{(1-p)w(t)}\int_a^t E^1_{r,1}[-\lambda^{\ln(\bar{p})}_{p,1}(t-\tau)^r](wf)'(\tau)d\tau.$$

12. When $\mu = q$, $\sigma = \delta = 1$, $m = r = p$, $w(t) = 1$ and $\phi(t) = t$, we have the fractional derivative introduced in [25] given by

$$^{C}D^{p,q,p,p,q}_{a,1,1,1,t}f(t) = \frac{H(p+q-1)}{2-p-q}\int_a^t (t-\tau)^{q-1}E^1_{p,q}[-\lambda^1_{p,q}(t-\tau)^p]f'(\tau)d\tau.$$

13. When $\mu = 2-q$, $\sigma = \delta = 1$, $m = r = p$, $w(t) = 1$ and $\phi(t) = t$, we have the modified fractional derivative [26] given by

$$^{C}D^{p,q,p,p,2-q}_{a,1,1,1,t}f(t) = \frac{H(p+q-1)}{2-p-q}\int_a^t (t-\tau)^{1-q}E^1_{p,2-q}[-\lambda^1_{p,q}(t-\tau)^p]f'(\tau)d\tau.$$

14. When $\mu = q = 1-p$, $w(t) = 1$ and $\phi(t) = t$, we have the Caputo fractional derivative [7] with singular kernel given by

$$^{C}D^{p,1-p,r,m,1-p}_{a,\sigma,\delta,1,t}f(t) = \frac{1}{\Gamma(1-p)}\int_a^t (t-\tau)^{-p}f'(\tau)d\tau.$$

15. When $\mu = q = 1-p$, $w(t) = 1$ and $\phi(t) = \ln(t)$, we have the Hadamard fractional derivative [10, 11] with singular kernel given by

$$ {}^{C}D^{p,1-p,r,m,1-p}_{a,\sigma,\delta,1,\ln(t)} f(t) = \frac{1}{\Gamma(1-p)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{-p} f'(\tau) d\tau. $$

16. When $\mu = q = 1-p$, $w(t) = 1$ and $\phi(t) = \frac{t^\rho}{\rho}$ with $\rho > 0$, we have the Katugampola fractional derivative fractional derivative [12] with singular kernel given by

$$ {}^{C}D^{p,1-p,r,m,1-p}_{a,\sigma,\delta,1,\frac{t^\rho}{\rho}} f(t) = \frac{\rho^p}{\Gamma(1-p)} \int_a^t \left(t^\rho - \tau^\rho\right)^{-p} f'(\tau) d\tau. $$

Now, we introduce the new generalized mixed fractional derivative in the Riemann-Liouville sense.

Definition 2.3. Let $(p, q) \in [0, 1]^2$, $r, m > 0$, $Re(\mu) > 0$, $\sigma \in \mathbb{R}$, $\delta \in \mathbb{R}^*$ and $f \in \mathcal{H}^1(a, b)$. The generalized mixed fractional derivative of the function $f(t)$ of order p in Riemann-Liouville sense with the weight function $w(t)$ and respect to another function $\phi(t)$ is given by

$$ {}^{R}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi} f(t) = \frac{\widetilde{H}(p,q)}{w(t)\phi'(t)} \frac{d}{dt} \int_a^t \phi'(\tau)\big(\phi(t)-\phi(\tau)\big)^{\mu-1} E^{\sigma}_{r,\mu}[-\lambda^{\delta}_{p,q}\big(\phi(t)-\phi(\tau)\big)^m](wf)(\tau)d\tau, \tag{2.2} $$

where $\widetilde{H}(p, q) = \frac{H(p+q-1)}{2-p-q}$.

Definition 2.3 includes all versions in the sense of Riemann-Liouville of the fractional derivatives with singular and non-singular kernels mentioned in Remark 2.2. It also includes the Riemann-Liouville fractional derivative [8, 9] when $\mu = q = 1-p$, $w(t) = 1$ and $\phi(t) = t$.

Lemma 2.4. *The generalized mixed fractional derivatives in the sense of Caputo and Riemann-Liouville can be expressed as follows:*

$$ \begin{aligned} {}^{R}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi} f(t) &= \widetilde{H}(p,q) \sum_{k=0}^{+\infty} \frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k \Gamma(km+\mu)}{k!\Gamma(kr+\mu)} \, {}^{R}\mathcal{J}^{km+\mu-1}_{a,w,\phi} f(t), \\ {}^{C}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi} f(t) &= \widetilde{H}(p,q) \sum_{k=0}^{+\infty} \frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k \Gamma(km+\mu)}{k!\Gamma(kr+\mu)} \, {}^{R}\mathcal{J}^{km+\mu}_{a,w,\phi} \left(\frac{(wf)'}{w\phi'}\right)(t), \end{aligned} $$

where ${}^{R}\mathcal{J}^{\alpha}_{a,w,\phi} f(t)$ *is the weighted Riemann-Liouville fractional integral of function* $f(t)$ *with respect to another* $\phi(t)$ *[27], which is given by*

$$ {}^{R}\mathcal{J}^{\alpha}_{a,w,\phi} f(t) = \frac{1}{\Gamma(\alpha)w(t)} \int_a^t \phi'(\tau)\big(\phi(t)-\phi(\tau)\big)^{\alpha-1}(wf)(\tau)d\tau. \tag{2.3} $$

Proof. Since the generalized Mittag-Leffler function $E^{\sigma}_{r,\mu}(t)$ is a locally uniformly convergent series on the entire complex plane, we have

$$ \begin{aligned} {}^{R}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi} f(t) &= \frac{\widetilde{H}(p,q)}{w(t)\phi'(t)} \sum_{k=0}^{+\infty} \frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k}{k!\Gamma(kr+\mu)} \frac{d}{dt}\left(w(t)\, {}^{R}\mathcal{J}^{km+\mu}_{a,w,\phi} f(t)\right) \\ &= \widetilde{H}(p,q) \sum_{k=0}^{+\infty} \frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k \Gamma(km+\mu)}{k!\Gamma(kr+\mu)} \, {}^{R}\mathcal{J}^{km+\mu-1}_{a,w,\phi} f(t). \end{aligned} $$

Similarly, we have

$$
\begin{aligned}
{}^{C}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t) &= \frac{\widetilde{H}(p,q)}{w(t)}\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k}{k!\Gamma(kr+\mu)}\int_a^t\big(\phi(t)-\phi(\tau)\big)^{km+\mu-1}(wf)'(\tau)d\tau \\
&= \widetilde{H}(p,q)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k\Gamma(km+\mu)}{k!\Gamma(kr+\mu)}\, {}^{R}\mathcal{J}^{km+\mu}_{a,w,\phi}\left(\frac{(wf)'}{w\phi'}\right)(t).
\end{aligned}
$$

This completes the proof. □

Remark 2.5. Lemma 2.4 extends the recent result presented in Proposition 3.5 of [32], it suffices to take $q=\delta=1$.

As in [27], the weighted Laplace transform of the function f with respect to another function ϕ is defined as follows:

$$
\mathcal{L}_{w,\phi}\{f(t)\}(s)=\int_a^{+\infty}\phi'(t)e^{-s\left(\phi(t)-\phi(a)\right)}w(t)f(t)dt. \tag{2.4}
$$

If $w(t)=1$ and $\phi(t)=t$, then (2.4) reduced to the classical Laplace transform. Also, it follows from [27] that

$$
\mathcal{L}_{w,\phi}\{{}^{R}\mathcal{J}^{\alpha}_{a,w,\phi}f(t)\}(s)=s^{-\alpha}\mathcal{L}_{w,\phi}\{f(t)\}(s). \tag{2.5}
$$

Theorem 2.6. *The weighted Laplace transforms with respect to another function ϕ of the generalized mixed fractional derivatives are given by*

$$
\begin{aligned}
\mathcal{L}_{w,\phi}\{{}^{R}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t)\}(s) &= \widetilde{H}(p,q)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k\Gamma(km+\mu)}{k!\Gamma(kr+\mu)s^{km+\mu-1}}\,\mathcal{L}_{w,\phi}\{f(t)\}(s), \\
\mathcal{L}_{w,\phi}\{{}^{C}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t)\}(s) &= \widetilde{H}(p,q)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k\Gamma(km+\mu)}{k!\Gamma(kr+\mu)s^{km+\mu}}\,[s\mathcal{L}_{w,\phi}\{f(t)\}(s)-(wf)(a)].
\end{aligned}
$$

Proof. According to Lemma 2.4 and (2.5), we have

$$
\begin{aligned}
\mathcal{L}_{w,\phi}\{{}^{R}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t)\}(s) &= \widetilde{H}(p,q)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k\Gamma(km+\mu)}{k!\Gamma(kr+\mu)}\,\mathcal{L}_{w,\phi}\{{}^{R}\mathcal{J}^{km+\mu-1}_{a,w,\phi}f(t)\}(s) \\
&= \widetilde{H}(p,q)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k\Gamma(km+\mu)}{k!\Gamma(kr+\mu)s^{km+\mu-1}}\,\mathcal{L}_{w,\phi}\{f(t)\}(s).
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
&\mathcal{L}_{w,\phi}\{{}^{C}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t)\} \\
&= \widetilde{H}(p,q)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k\Gamma(km+\mu)}{k!\Gamma(kr+\mu)}\,\mathcal{L}_{w,\phi}\{{}^{R}\mathcal{J}^{km+\mu}_{a,w,\phi}\left(\frac{(wf)'}{w\phi'}\right)\}(t) \\
&= \widetilde{H}(p,q)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k\Gamma(km+\mu)}{k!\Gamma(kr+\mu)\Gamma(kr+\mu)s^{km+\mu}}\int_a^{+\infty}e^{-s\left(\phi(t)-\phi(a)\right)}(wf)'(t)dt \\
&= \widetilde{H}(p,q)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k\Gamma(km+\mu)}{k!\Gamma(kr+\mu)s^{km+\mu}}\,[s\mathcal{L}_{w,\phi}\{f(t)\}(s)-(wf)(a)].
\end{aligned}
$$

This ends the proof.

Theorem 2.7. *The relation between both generalized mixed fractional derivatives is given by*

$$\begin{aligned} {}^{R}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t) &= {}^{C}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t) \\ &+ \frac{H(p+q-1)\big(\phi(t)-\phi(a)\big)^{\mu-1}}{(2-p-q)w(t)}E^{\sigma}_{r,\mu}[-\lambda^{\delta}_{p,q}\big(\phi(t)-\phi(a)\big)^{m}](wf)(a). \end{aligned} \tag{2.6}$$

Proof. We have

$$\begin{aligned} &\mathcal{L}_{w,\phi}\{{}^{C}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t)\}(s) \\ =\ & \widetilde{H}(p,q)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k\Gamma(km+\mu)}{k!\Gamma(kr+\mu)s^{km+\mu}}\ [s\mathcal{L}_{w,\phi}\{f(t)\}(s)-(wf)(a)] \\ =\ & \widetilde{H}(p,q)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k\Gamma(km+\mu)}{k!\Gamma(kr+\mu)s^{km+\mu-1}}\ \mathcal{L}_{w,\phi}\{f(t)\}(s) \\ & -\widetilde{H}(p,q)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k\Gamma(km+\mu)}{k!\Gamma(kr+\mu)s^{km+\mu}}\ (wf)(a) \\ =\ & \mathcal{L}_{w,\phi}\{{}^{R}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t)\}(s) \\ & -\widetilde{H}(p,q)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k\Gamma(km+\mu)}{k!\Gamma(kr+\mu)s^{km+\mu}}\ (wf)(a) \\ =\ & \mathcal{L}_{w,\phi}\{{}^{R}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t)\}(s) \\ & -\widetilde{H}(p,q)(wf)(a)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k}{k!\Gamma(kr+\mu)}\ \mathcal{L}_{w,\phi}\{\frac{\big(\phi(t)-\phi(a)\big)^{km+\mu-1}}{w(t)}\}(s). \end{aligned}$$

By applying the inverse Laplace, we get

$$\begin{aligned} &{}^{C}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t) \\ =\ & {}^{R}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t)-\widetilde{H}(p,q)(wf)(a)\sum_{k=0}^{+\infty}\frac{(\sigma)_k(-\lambda^{\delta}_{p,q})^k}{k!\Gamma(kr+\mu)}\ \frac{\big(\phi(t)-\phi(a)\big)^{km+\mu-1}}{w(t)} \\ =\ & {}^{R}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t) \\ & -\frac{H(p+q-1)\big(\phi(t)-\phi(a)\big)^{\mu-1}}{(2-p-q)w(t)}E^{\sigma}_{r,\mu}[-\lambda^{\delta}_{p,q}\big(\phi(t)-\phi(a)\big)^{m}](wf)(a). \end{aligned}$$

Hence,

$$\begin{aligned} {}^{R}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t) &= {}^{C}D^{p,q,r,m,\mu}_{a,\sigma,\delta,w,\phi}f(t) \\ &+ \frac{H(p+q-1)\big(\phi(t)-\phi(a)\big)^{\mu-1}}{(2-p-q)w(t)}E^{\sigma}_{r,\mu}[-\lambda^{\delta}_{p,q}\big(\phi(t)-\phi(a)\big)^{m}](wf)(a). \end{aligned}$$

This completes the proof. □

Remark 2.8. Theorems 2.6 and 2.7 extend the recent results established in [32] for $q = \delta = 1$, the results in [25] for $\mu = q$, $\sigma = \delta = 1$, $m = r = p$, $w(t) = 1$ and $\phi(t) = t$, the results in [26] for $\mu = 2 - q$, $\sigma = \delta = 1$, $m = r = p$, $w(t) = 1$ and $\phi(t) = t$, as well as the results in [23] for $\mu = q$, $\sigma = 1$ and $\phi(t) = t$.

3. The new associate fractional integral

This section defines the generalized fractional integral associated to the new class of fractional derivatives. First, we need the following lemma.

Lemma 3.1. *Let* $(p, q) \in [0, 1]^2$, $m = r > 0$, $\mathrm{Re}(\mu) > 0$, $\delta \in \mathbb{R}^*$ *and* $\sigma \in \mathbb{N}^*$. *The following fractional differential equation:*

$$^{R}D^{p,q,r,r,\mu}_{a,\sigma,\delta,w,\phi}\, y(t) = f(t), \tag{3.1}$$

has a unique solution given by

$$y(t) = \sum_{k=0}^{\sigma} \binom{\sigma}{k} \frac{\delta^k (p+q-1)^k}{(2-p-q)^{k-1} H(p+q-1)}\, {}^{R}\mathcal{J}^{kr-\mu+1}_{a,w,\phi} f(t). \tag{3.2}$$

Proof. From (3.1) and Theorem 2.6, we have

$$\frac{H(p+q-1)}{2-p-q} \sum_{k=0}^{+\infty} \frac{(\sigma)_k (-\lambda^{\delta}_{p,q})^k}{k!\, s^{kr+\mu-1}}\, \mathcal{L}_{w,\phi}\{y(t)\}(s) = \mathcal{L}_{w,\phi}\{f(t)\}(s). \tag{3.3}$$

For $\sigma = 1$, (3.3) becomes

$$\frac{H(p+q-1)}{2-p-q} \sum_{k=0}^{+\infty} \frac{(-\lambda^{\delta}_{p,q})^k}{s^{kr+\mu-1}}\, \mathcal{L}_{w,\phi}\{y(t)\}(s) = \mathcal{L}_{w,\phi}\{f(t)\}(s).$$

Hence,

$$\begin{aligned}
\mathcal{L}_{w,\phi}\{y(t)\}(s) &= \frac{2-p-q}{H(p+q-1)} s^{\mu-1} (1 + \lambda^{\delta}_{p,q} s^{-r}) \mathcal{L}_{w,\phi}\{f(t)\}(s) \\
&= \frac{2-p-q}{H(p+q-1)} s^{\mu-1} \mathcal{L}_{w,\phi}\{f(t)\}(s) + \frac{\delta(p+q-1)}{H(p+q-1)} s^{\mu-r-1} \mathcal{L}_{w,\phi}\{f(t)\}(s), \\
&= \frac{2-p-q}{H(p+q-1)} \mathcal{L}_{w,\phi}\{{}^{R}\mathcal{J}^{1-\mu}_{a,w,\phi} f(t)\}(s) + \frac{\delta(p+q-1)}{H(p+q-1)} \mathcal{L}_{w,\phi}\{{}^{R}\mathcal{J}^{1-\mu+r}_{a,w,\phi} f(t)\}(s).
\end{aligned}$$

By passage to the inverse Laplace, we obtain

$$y(t) = \frac{2-p-q}{H(p+q-1)}\, {}^{R}\mathcal{J}^{1-\mu}_{a,w,\phi} f(t) + \frac{\delta(p+q-1)}{H(p+q-1)}\, {}^{R}\mathcal{J}^{1+r-\mu}_{a,w,\phi} f(t).$$

For $\sigma = 2$, (3.3) becomes

$$\frac{H(p+q-1)}{2-p-q} \sum_{k=0}^{+\infty} \frac{(k+1)(-\lambda^{\delta}_{p,q})^k}{s^{kr+\mu-1}}\, \mathcal{L}_{w,\phi}\{y(t)\}(s) = \mathcal{L}_{w,\phi}\{f(t)\}(s).$$

Hence,

$$\begin{aligned}
\mathcal{L}_{w,\phi}\{y(t)\}(s) &= \frac{2-p-q}{H(p+q-1)} s^{\mu-1}(1+\lambda^{\delta}_{p,q} s^{-r})^2 \mathcal{L}_{w,\phi}\{f(t)\}(s),\\
&= \frac{2-p-q}{H(p+q-1)} s^{\mu-1}\mathcal{L}_{w,\phi}\{f(t)\}(s) + \frac{2\delta(p+q-1)}{H(p+q-1)} s^{\mu-r-1}\mathcal{L}_{w,\phi}\{f(t)\}(s)\\
&\quad + \frac{\delta^2(p+q-1)^2}{(2-p-q)H(p+q-1)} s^{\mu-2r-1}\mathcal{L}_{w,\phi}\{f(t)\}(s),\\
&= \frac{2-p-q}{H(p+q-1)} \mathcal{L}_{w,\phi}\{{}^{R}\mathcal{J}^{1-\mu}_{a,w,\phi} f(t)\}(s) + \frac{2\delta(p+q-1)}{H(p+q-1)} \mathcal{L}_{w,\phi}\{{}^{R}\mathcal{J}^{1-\mu+r}_{a,w,\phi} f(t)\}(s)\\
&\quad + \frac{\delta^2(p+q-1)^2}{(2-p-q)H(p+q-1)} \mathcal{L}_{w,\phi}\{{}^{R}\mathcal{J}^{1-\mu+2r}_{a,w,\phi} f(t)\}(s).
\end{aligned}$$

By applying the inverse Laplace transform, we have

$$\begin{aligned}
y(t) &= \frac{2-p-q}{H(p+q-1)} {}^{R}\mathcal{J}^{1-\mu}_{a,w,\phi} f(t) + \frac{2\delta(p+q-1)}{H(p+q-1)} {}^{R}\mathcal{J}^{1-\mu+r}_{a,w,\phi} f(t)\\
&\quad + \frac{\delta^2(p+q-1)^2}{(2-p-q)H(p+q-1)} {}^{R}\mathcal{J}^{1-\mu+2r}_{a,w,\phi} f(t).
\end{aligned}$$

Similarly, when $\sigma = n$, we have

$$y(t) = \sum_{k=0}^{\sigma} \binom{\sigma}{k} \frac{\delta^k (p+q-1)^k}{(2-p-q)^{k-1} H(p+q-1)} {}^{R}\mathcal{J}^{kr-\mu+1}_{a,w,\phi} f(t),$$

which ends the proof. □

Definition 3.2. If $m = r$, then the fractional integral associated to the generalized mixed fractional derivative is defined as follows

$$I^{p,q,r,\mu}_{a,\sigma,\delta,w,\phi} f(t) = \sum_{k=0}^{+\infty} \binom{\sigma}{k} \frac{\delta^k (p+q-1)^k}{(2-p-q)^{k-1} H(p+q-1)} {}^{R}\mathcal{J}^{kr-\mu+1}_{a,w,\phi} f(t). \tag{3.4}$$

Remark 3.3. Definition 3.2 includes many forms of fractional integrals existing in the literature. More precisely,

1. When $q = \delta = 1$ in Eq. (3.4), we have the generalized weighted fractional integral with respect to another function [32] given by

$$I^{p,1,r,\mu}_{a,\sigma,1,w,\phi} f(t) = \sum_{k=0}^{+\infty} \binom{\sigma}{k} \frac{p^k}{(1-p)^{k-1} H(p)} {}^{R}\mathcal{J}^{kr-\mu+1}_{a,w,\phi} f(t).$$

2. When $q = \delta = \mu = \sigma = 1$ and $\phi(t) = t$ in Eq. (3.4), we have the GHF integral [16] given by

$$I^{p,1,r,1}_{a,1,1,w,t} f(t) = \frac{1-p}{H(p)} f(t) + \frac{p}{H(p)} {}^{R}\mathcal{J}^{r}_{a,w,t} f(t).$$

3. When $r = p$, $q = \delta = 1$, $w(t) = 1$ and $\phi(t) = t$ in Eq. (3.4), we have the fractional integral corresponding to the generalized AB fractional derivative with the generalized Mittag-Leffler function [21] given by

$$I^{p,1,p,\mu}_{a,\sigma,1,1,t} f(t) = \sum_{k=0}^{+\infty} \binom{\sigma}{k} \frac{p^k}{(1-p)^{k-1} H(p)} {}^{R}\mathcal{J}^{kr-\mu+1}_{a,1,t} f(t).$$

4. When $r = p$, $q = \delta = \sigma = 1$, $w(t) = 1$ and $\phi(t) = t$, we have the fractional integral corresponding to the generalized AB fractional derivative with the generalized Mittag-Leffler function [27] given by

$$I^{p,1,p,\mu}_{a,1,1,1,t} f(t) = \frac{1-p}{H(p)} f(t) + \frac{p}{H(p)} {}^{R}\mathcal{J}^{p-\mu+1}_{a,w,t} f(t).$$

5. When $r = p$, $q = \delta = \mu = \sigma = 1$ and $\phi(t) = t$ in Eq. (3.4), we have the weighted AB fractional integral [15] given by

$$I^{p,1,p,1}_{a,1,1,w,t} f(t) = \frac{1-p}{H(p)} f(t) + \frac{p}{H(p)} {}^{R}\mathcal{J}^{p}_{a,w,t} f(t).$$

6. When $r = p$, $q = \delta = \mu = \sigma = 1$ and $w(t) = 1$ in Eq. (3.4), we have the AB fractional integral with respect to another function [22] given by

$$I^{p,1,p,1}_{a,1,1,1,\phi} f(t) = \frac{1-p}{H(p)} f(t) + \frac{p}{H(p)} {}^{R}\mathcal{J}^{p}_{a,w,\phi} f(t).$$

7. When $r = p$, $q = \delta = \mu = \sigma = 1$, $w(t) = 1$ and $\phi(t) = t$ in Eq. (3.4), we have the AB fractional integral [14] given by

$$I^{p,1,p,1}_{a,1,1,1,t} f(t) = \frac{1-p}{H(p)} f(t) + \frac{p}{H(p)} {}^{R}\mathcal{J}^{p}_{a,1,t} f(t).$$

8. When $r = q = \delta = \mu = \sigma = 1$ in Eq. (3.4), we have the weighted CF fractional integral with respect to another function [17] given by

$$I^{p,1,1,1}_{a,1,1,w,\phi} f(t) = \frac{1-p}{H(p)} f(t) + \frac{p}{H(p)} \frac{1}{w(t)} \int_a^t \phi'(\tau) w(\tau) f(\tau) d\tau.$$

9. When $r = q = \delta = \mu = \sigma = 1$ and $w(t) = 1$ in Eq. (3.4), we have the CF fractional integral with respect to another function [17] given by

$$I^{p,1,1,1}_{a,1,1,1,\phi} f(t) = \frac{1-p}{H(p)} f(t) + \frac{p}{H(p)} \int_a^t \phi'(\tau) f(\tau) d\tau.$$

10. When $r = q = \delta = \mu = \sigma = 1$, $w(t) = 1$ and $\phi(t) = t$ in Eq. (3.4), we have the CF fractional derivative [13] given by

$$I^{p,1,1,1}_{a,1,1,1,t} f(t) = \frac{1-p}{H(p)} f(t) + \frac{p}{H(p)} \int_a^t f(\tau) d\tau.$$

11. When $\mu = q$, $\sigma = 1$ and $\phi(t) = t$ in Eq. (3.4), we have the fractional integral corresponding to the new mixed fractional derivative [23] given by

$$I^{p,q,r,q}_{a,1,\delta,w,t} f(t) = \frac{2-p-q}{H(p+q-1)} {}^{R}\mathcal{J}^{1-q}_{a,w,t} f(t) + \frac{\delta(p+q-1)}{H(p+q-1)} {}^{R}\mathcal{J}^{r-q+1}_{a,w,t} f(t).$$

12. When $\mu = q = \sigma = 1$, $\delta = \ln(\bar{p})$ (with $\bar{p} > 0$) and $\phi(t) = t$ in Eq. (3.4), we have the fractional integral [24] given by

$$I^{p,1,r,1}_{a,1,\ln(\bar{p}),w,t} f(t) = \frac{1-p}{H(p)} f(t) + \frac{p\ln(\bar{p})}{H(p)} {}^{R}\mathcal{J}^{r}_{a,w,t} f(t).$$

13. When $\mu = q$, $\sigma = \delta = 1$, $r = p$, $w(t) = 1$ and $\phi(t) = t$ in Eq. (3.4), we have the fractional integral introduced in [25] given by

$$I^{p,q,p,q}_{a,1,1,1,t} f(t) = \frac{2-p-q}{H(p+q-1)} {}^{R}\mathcal{J}^{1-q}_{a,1,t} f(t) + \frac{p+q-1}{H(p+q-1)} {}^{R}\mathcal{J}^{p-q+1}_{a,1,t} f(t).$$

14. When $\mu = 2-q$, $\sigma = \delta = 1$, $r = p$, $w(t) = 1$ and $\phi(t) = t$ in Eq. (3.4), we have the modified fractional integral [26] given by

$$I^{p,q,p,q}_{a,1,1,1,t} f(t) = \frac{2-p-q}{H(p+q-1)} {}^{R}\mathcal{J}^{q-1}_{a,1,t} f(t) + \frac{p+q-1}{H(p+q-1)} {}^{R}\mathcal{J}^{p+q-1}_{a,1,t} f(t).$$

15. When $\mu = q = 1-p$ in Eq. (3.4), we have the weighted Riemann-Liouville fractional integral with respect to another [27] given by

$$I^{p,1-p,r,1-p}_{a,\sigma,\delta,w,\phi} f(t) = \frac{1}{\Gamma(p)w(t)} \int_a^t \phi'(\tau)\big(\phi(t)-\phi(\tau)\big)^{p-1}(wf)(\tau)d\tau.$$

16. When $\mu = q = 1-p$, $w(t) = 1$ and $\phi(t) = \ln(t)$ in Eq. (3.4), we have the Hadamard fractional integral [18, 19] given by

$$I^{p,1-p,r,1-p}_{a,\sigma,\delta,1,\ln(t)} f(t) = \frac{1}{\Gamma(p)} \int_a^t \Big(\ln\frac{t}{\tau}\Big)^{p-1}\frac{f(\tau)}{\tau}d\tau.$$

17. When $\mu = q = 1-p$, $w(t) = 1$ and $\phi(t) = \frac{t^\rho}{\rho}$ with $\rho > 0$ in Eq. (3.4), we have the Katugampola fractional integral [20] given by

$$I^{p,1-p,r,1-p}_{a,\sigma,\delta,1,\frac{t^\rho}{\rho}} f(t) = \frac{\rho^{1-p}}{\Gamma(p)} \int_a^t \big(t^\rho-\tau^\rho\big)^{p-1}\frac{f(\tau)}{\tau^{1-\rho}}d\tau.$$

18. When $\mu = q = 1-p$, $w(t) = 1$ and $\phi(t) = t$ in Eq. (3.4), we have the Riemann-Liouville fractional integral [8] given by

$$I^{p,1-p,r,1-p}_{a,\sigma,\delta,1,\frac{t^\rho}{\rho}} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau)d\tau.$$

19. When $\mu = q = 1-p$ and $w(t) = 1$ in Eq. (3.4), we have the Riemann-Liouville fractional integral with respect to another function [28, 29, 8] given by

$$I^{p,1-p,r,1-p}_{a,\sigma,\delta,1,\phi} f(t) = \frac{1}{\Gamma(p)} \int_a^t \phi'(\tau)\big(\phi(t)-\phi(\tau)\big)^{p-1} f(\tau)d\tau.$$

20. When $\mu = q = 1-p$ and $w(t) = e^{\alpha t}$ and $\phi(t) = t$ in Eq. (3.4), we have the tempered fractional integral [30, 31] given by

$$I^{p,1-p,r,1-p}_{a,\sigma,\delta,e^{\alpha t},t} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} e^{-\alpha(t-\tau)} f(\tau)d\tau.$$

4. Fundamental properties

This section presents some fundamental properties for the new generalized class of fractional operators of differentiation and integration.

Theorem 4.1. *The new generalized mixed fractional derivative and integral satisfy the the following properties:*

(i) $I^{p,q,r,\mu}_{a,\sigma,\delta,w,\phi}\left({}^{R}D^{p,q,r,r,\mu}_{a,\sigma,\delta,w,\phi}f\right)(t)=f(t)$.

(ii) ${}^{R}D^{p,q,r,r,\mu}_{a,\sigma,\delta,w,\phi}\left(I^{p,q,r,\mu}_{a,\sigma,\delta,w,\phi}f\right)(t)=f(t)$.

(iii) $I^{p,q,r,\mu}_{a,\sigma,\delta,w,\phi}\left({}^{C}D^{p,q,r,r,\mu}_{a,\sigma,\delta,w,\phi}f\right)(t)=f(t)-\dfrac{w(a)f(a)}{w(t)}$.

Proof. From (3.4), we have

$$I^{p,q,r,\mu}_{a,\sigma,\delta,w,\phi}\left({}^{R}D^{p,q,r,r,\mu}_{a,\sigma,\delta,w,\phi}f\right)(t)=\sum_{i=0}^{+\infty}\binom{\sigma}{i}\frac{\delta^{i}(p+q-1)^{i}}{(2-p-q)^{i-1}H(p+q-1)}\,{}^{R}\mathcal{J}^{ir-\mu+1}_{a,w,\phi}\left({}^{R}D^{p,q,r,r,\mu}_{a,\sigma,\delta,w,\phi}f\right)(t).$$

According to Lemma 2.4, we get

$$\begin{aligned}
I^{p,q,r,\mu}_{a,\sigma,\delta,w,\phi}\left({}^{R}D^{p,q,r,r,\mu}_{a,\sigma,\delta,w,\phi}f\right)(t) &= \sum_{i=0}^{+\infty}\sum_{k=0}^{+\infty}\binom{\sigma}{i}\frac{\delta^{i}(p+q-1)^{i}(\sigma)_{k}(-\lambda^{\delta}_{p,q})^{k}}{(2-p-q)^{i}k!}\,{}^{R}\mathcal{J}^{(i+k)r}_{a,w,\phi}f(t)\\
&= \sum_{i=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^{i}\binom{\sigma}{i}(\sigma)_{k}(-\lambda^{\delta}_{p,q})^{i+k}}{k!}\,{}^{R}\mathcal{J}^{(i+k)r}_{a,w,\phi}f(t)\\
&= \sum_{m=0}^{+\infty}(-\lambda^{\delta}_{p,q})^{m}\,{}^{R}\mathcal{J}^{mr}_{a,w,\phi}f(t)\sum_{i=0}^{m}\frac{(-1)^{i}\binom{\sigma}{i}(\sigma)_{m-i}}{(m-i)!}\\
&= f(t)+\sum_{m=1}^{+\infty}(-\lambda^{\delta}_{p,q})^{m}\,{}^{R}\mathcal{J}^{mr}_{a,w,\phi}f(t)\sum_{i=0}^{m}\frac{(-1)^{i}\binom{\sigma}{i}(\sigma)_{m-i}}{(m-i)!}.
\end{aligned}$$

Since $\sum_{i=0}^{m}\frac{(-1)^{i}\binom{\sigma}{i}(\sigma)_{m-i}}{(m-i)!}=0$ (see, Lemma 3 of [34]), we deduce (i).
For (ii), we have

$$\begin{aligned}
{}^{R}D^{p,q,r,r,\mu}_{a,\sigma,\delta,w,\phi}\left(I^{p,q,r,\mu}_{a,\sigma,\delta,w,\phi}f\right)(t) &= {}^{R}D^{p,q,r,r,\mu}_{a,\sigma,\delta,w,\phi}\Big[\sum_{i=0}^{+\infty}\frac{\binom{\sigma}{i}\delta^{i}(p+q-1)^{i}}{(2-p-q)^{i-1}H(p+q-1)}\,{}^{R}\mathcal{J}^{ir-\mu+1}_{a,w,\phi}f(t)\Big]\\
&= \sum_{k=0}^{+\infty}\sum_{i=0}^{+\infty}\frac{(-1)^{i}\binom{\sigma}{i}(\sigma)_{k}(-\lambda^{\delta}_{p,q})^{i+k}}{k!}\,{}^{R}\mathcal{J}^{(i+k)r}_{a,w,\phi}f(t)\\
&= \sum_{m=0}^{+\infty}(-\lambda^{\delta}_{p,q})^{m}\,{}^{R}\mathcal{J}^{mr}_{a,w,\phi}f(t)\sum_{i=0}^{m}\frac{(-1)^{i}\binom{\sigma}{i}(\sigma)_{m-i}}{(m-i)!}\\
&= f(t)+\sum_{m=1}^{+\infty}(-\lambda^{\delta}_{p,q})^{m}\,{}^{R}\mathcal{J}^{mr}_{a,w,\phi}f(t)\sum_{i=0}^{m}\frac{(-1)^{i}\binom{\sigma}{i}(\sigma)_{m-i}}{(m-i)!}\\
&= f(t).
\end{aligned}$$

For (iii), we have

$$
\begin{aligned}
& I^{p,q,r,\mu}_{a,\sigma,\delta,w,\phi}\left({}^{C}D^{p,q,r,r,\mu}_{a,\sigma,\delta,w,\phi}f\right)(t) \\
= & \sum_{i=0}^{+\infty}\binom{\sigma}{i}\frac{\delta^i(p+q-1)^i}{(2-p-q)^{i-1}H(p+q-1)}\,{}^{R}\mathcal{J}^{ir-\mu+1}_{a,w,\phi}\left({}^{C}D^{p,q,r,r,\mu}_{a,\sigma,\delta,w,\phi}f\right)(t) \\
= & \sum_{i=0}^{+\infty}\sum_{k=0}^{+\infty}\binom{\sigma}{i}\frac{\delta^i(p+q-1)^i(\sigma)_k(-\lambda^{\delta}_{p,q})^k}{(2-p-q)^i k!}\,{}^{R}\mathcal{J}^{(i+k)r+1}_{a,w,\phi}\left(\frac{(wf)'}{w\phi'}\right)(t) \\
= & \sum_{i=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{(-1)^i\binom{\sigma}{i}(\sigma)_k(-\lambda^{\delta}_{p,q})^{i+k}}{k!}\,{}^{R}\mathcal{J}^{(i+k)r+1}_{a,w,\phi}\left(\frac{(wf)'}{w\phi'}\right)(t) \\
= & \sum_{m=0}^{+\infty}(-\lambda^{\delta}_{p,q})^m\,{}^{R}\mathcal{J}^{mr+1}_{a,w,\phi}\left(\frac{(wf)'}{w\phi'}\right)(t)\sum_{i=0}^{m}\frac{(-1)^i\binom{\sigma}{i}(\sigma)_{m-i}}{(m-i)!}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I^{p,q,r,\mu}_{a,\sigma,\delta,w,\phi}\left({}^{C}D^{p,q,r,r,\mu}_{a,\sigma,\delta,w,\phi}f\right)(t) &= {}^{R}\mathcal{J}^{1}_{a,w,\phi}\left(\frac{(wf)'}{w\phi'}\right)(t) \\
&\quad + \sum_{m=1}^{+\infty}(-\lambda^{\delta}_{p,q})^m\,{}^{R}\mathcal{J}^{mr+1}_{a,w,\phi}\left(\frac{(wf)'}{w\phi'}\right)(t)\sum_{i=0}^{m}\frac{(-1)^i\binom{\sigma}{i}(\sigma)_{m-i}}{(m-i)!} \\
&= \frac{1}{w(t)}\int_a^t (wf)'(\tau)d\tau \\
&= f(t)-\frac{w(a)f(a)}{w(t)}.
\end{aligned}
$$

This completes the proof of theorem. □

Clearly, (iii) of Theorem 4.1 extends the Newton-Leibniz formula established in [8] for Caputo fractional derivative with singular kernel, in [23] for mixed fractional derivative, and in [35] for AB fractional derivative in Caputo sence. In the fact, we have the following result.

Corollary 4.2. *The generalized mixed fractional derivative and integral satisfy the Newton-Leibniz formula. In other words, we have*

$$
I^{p,q,r,\mu}_{a,\sigma,\delta,1,\phi}\left({}^{C}D^{p,q,r,r,\mu}_{a,\sigma,\delta,1,\phi}f\right)(t) = f(t)-f(a). \tag{4.1}
$$

Obviously, ${}^{C}D^{p,q,r,r,\mu}_{a,\sigma,\delta,1,\phi}(C)=0$ for all constant function $f(t)=C$. In addition, we have the following result.

Corollary 4.3. *Let f be a solution of the following fractional differential equation*

$$
{}^{C}D^{p,q,r,r,\mu}_{a,\sigma,\delta,1,\phi}f(t)=0. \tag{4.2}
$$

Then the function f is a constant function.

Proof. According to (4.1) and (4.2), we deduce that $f(t)=f(a)$. This leads that f is a constant function. □

Now, consider the following simple initial value problem (IVP):

$$\begin{cases} {}^{C}D^{p,1,2,2,\frac{1}{2}}_{0,1,\delta,e^t,t}x(t) = e^{-t}, \\ x(0) = 0. \end{cases} \tag{4.3}$$

By applying the generalized fractional integral to both sides of (4.3) and using Theorem 4.1 (iii), we deduce that the IVP (4.3) has a unique solution given by

$$x(t) = \left(\frac{2(1-p)}{H(p)} + \frac{8p\delta t^2}{15H(p)}\right)\sqrt{\frac{t}{\pi}}e^{-t}. \tag{4.4}$$

For $H(p) = 1 - p + \dfrac{p}{\Gamma(p)}$, the impact of the parameters p and δ on the solution (4.4) is shown in Figures 1 and 2, respectively.

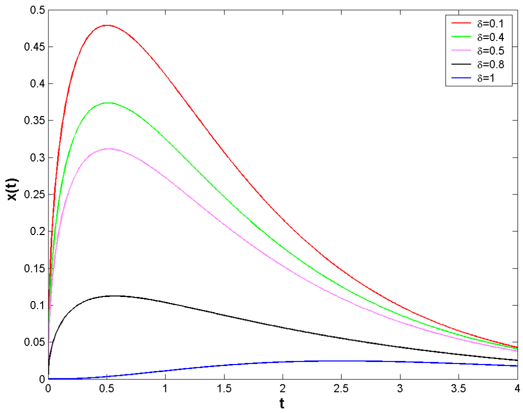

Figure 1: The solution of (4.3) with $\delta = 0.1$ for different values of p.

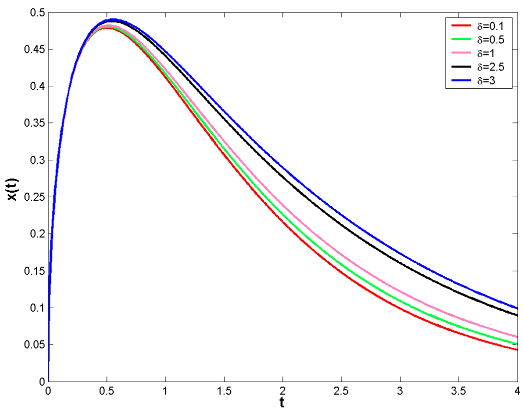

Figure 2: The solution of (4.3) with $p = 0.1$ for different values of δ.

5. Conclusion

In this study, we have defined a new fractional derivative in the sense of Caputo and Riemann-Liouville which generalizes numerous definitions of fractional derivatives with singular and non-singular kernels available in the classical and recent literature of fractional calculus. The corresponding fractional integral of the new generalized mixed fractional derivative has been derived by means of weighted Laplace transform with respect to another function. The novel fractional integral covers more than twenty definitions of various types of fractional integrals. Furthermore, we have established some new important formulas and fundamental properties of such new generalized class of fractional differential and integral operators. In particular, the Newton-Leibniz formula has been extended to include many special cases existing in the previous studies.

Acknowledgement

The author would like to thank the editor and anonymous referees for their valuable comments and suggestions that significantly improved the quality of this article.

References

[1] K. Shah, T. Abdeljawad, *On complex fractal-fractional order mathematical modeling of CO_2 emanations from energy sector*, Physica Scripta 99 (2024) 015226. https://doi.org/10.1088/1402-4896/ad1286

[2] Eiman, K. Shah, M. Sarwar, T. Abdeljawad, On rotavirus infectious disease model using piecewise modified ABC fractional order derivative, Networks and Heterogeneous Media 19 (1) (2024) 214–234. https://doi.org/10.3934/nhm.2024010

[3] Eiman, K. Shah, M. Sarwar, T. Abdeljawad, *On mathematical model of infectious disease by using fractals fractional analysis*, Discrete & Continuous Dynamical Systems - Series S 17 (10) (2024) 3064–3085. https://doi.org/10.3934/dcdss.2024073

[4] H. EL Mamouni, K. Hattaf, N. Yousfi, *Dynamics of an epidemic model for COVID-19 with Hattaf fractal-fractional operator and study of existence of solutions by means of fixed point theory*, Journal of Mathematics and Computer Science 36 (4) (2025) 371–385. https://doi.org/10.22436/jmcs.036.04.02

[5] H. EL Mamouni, K. Hattaf, N. Yousfi, *Dynamics of enzyme kinetic model under the new generalized Hattaf fractional derivative*, Mathematical Modeling and Computing 11 (2) (2024) 463–469. https://doi.org/10.23939/mmc2024.02.463

[6] S. Lasfar, M. Elkarmouchi, K. Hattaf, N. Yousfi, *Mathematical analysis of the dynamics of a fractional economic cycle model and the existence of solutions by means of fixed point theory*, 14 (2024) 1-13. https://doi.org/10.28919/afpt/8350

[7] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent-II*, Geophysical Journal International 13 (1967) 529–539. https://doi.org/10.1111/j.1365-246X.1967.tb02303.x

[8] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, "Theory and Applications of Fractional Differential Equations", North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006.

[9] I. Podlubny, "Fractional Differential Equations", Academic Press, San Diego, 1999.

[10] Y. Y Gambo, F. Jarad, D. Baleanu, T. Abdeljawad, *On Caputo modification of the Hadamard fractional derivatives*, Advances in Difference Equations (2014) 2014:10. https://doi.org/10.1186/1687-1847-2014-10

[11] F. Jarad, T. Abdeljawad and D. Baleanu, *Caputo-type modification of the Hadamard fractional derivatives*, Advances in Difference Equations (2012) 2012:142. https://doi.org/10.1186/1687-1847-2012-142

[12] R. Almeida, A. B. Malinowska, T. Odzijewicz, *Fractional differential equations with dependence on the Caputo-Katugampola derivative*, J. Comput. Nonlinear Dyn. 11(6) (2016) 061017. https://doi.org/10.1115/1.4034432

[13] M. Caputo, M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progress Fractional Differention and Applications (2015) 73–85. https://doi.org/10.12785/pfda/010201

[14] A. Atangana, D. Baleanu, *New fractional derivatives with non-local and non-singular kernel: theory and application to heat transfer model*, Thermal Science 20 (2016) 763–769. https://doi.org/10.2298/TSCI160111018A
[15] M. Al-Refai, *On weighted AtanganaBaleanu fractional operators*, Advances in Difference Equations 2020 (2020) 1–11. https://doi.org/10.1186/s13662-019-2471-z
[16] K. Hattaf, *A new generalized definition of fractional derivative with non-singular kernel*, Computation 8 (2) (2020) 1–9. https://doi.org/10.3390/computation8020049
[17] M. Al-Refai, A. Jarrah, *Fundamental results on weighted Caputo-Fabrizio fractional derivative*, Chaos, Solitons & Fractals 126 (2019) 7–11. https://doi.org/10.1016/j.chaos.2019.05.035
[18] J. Hadamard, *Essai sur l'étude des fonctions données par leur developpment de Taylor*, J. Math. Pures et Appl., Ser. (1892) 101–186.
[19] A. A. Kilbas, *Hadamard-type fractional calculus*, J. Korean Math. Soc. 38 (6) (2001) 1191–1204.
[20] U. N. Katugampola, *New approach to a generalized fractional integral*, Applied Mathematics and Computation 218 (2011) 860–865. https://doi.org/10.1016/j.amc.2011.03.062
[21] T. Abdeljawad, D. Baleanu, *On fractional derivatives with generalized Mittag-Leffler kernels*, Advances in Difference Equations (2018) 2018:468. https://doi.org/10.1186/s13662-018-1914-2
[22] A. Fernandez, D. Baleanu, *Differintegration with respect to functions in fractional models involving Mittag-Leffler functions*, SSRN Electron. J. (2018) 1–5. https://doi.org/10.2139/ssrn.3275746
[23] K. Hattaf, *A new mixed fractional derivative with applications in computational biology*, Computation 12 (1) (2024) 1–17. https://doi.org/10.3390/computation12010007
[24] E. M. Lotfi, H. Zine, D. F. M. Torres, N. Yousfi, *The power fractional calculus: First definitions and properties with applications to power fractional differential equations*, Mathematics 10 (2022) 1–10. https://doi.org/10.3390/math10193594
[25] S. M. Chinchole, A. P. Bhadane, *A new definition of fractional derivatives with Mittag-Leffler kernel of two parameters*, Communications in Mathematics and Applications 13 (1) (2022) 19–26. https://doi.org/10.26713/cma.v13i1.1689
[26] S. M. Chinchole, *Modified Definitions of SABC and SABR Fractional Derivatives and Applications*, International Journal of Difference Equations (IJDE) 17 (1) (2022) 87–112.
[27] F. Jarad, T. Abdeljawad, K. Shah, *On the weighted fractional operators of a function with respect to another function*, Fractals 28 (8) (2020) 1–12. https://doi.org/10.1142/S0218348X20400113
[28] T. J. Osler, *Leibniz rule for fractional derivatives generalized and an application to infinite series*, SIAM J. Appl. Math. 18 (1970) 658–674. https://doi.org/10.1137/0118059
[29] S. G. Samko, A. A. Kilbas, O. I. Marichev, "Fractional Integrals and Derivatives: Theory and Applications", Gordon and Breach Science New York, 1993.
[30] C. Li, W. Deng, L. Zhao, *Well-posedness and numerical algorithm for the tempered fractional ordinary differential equations*, Discrete Contin. Dyn. Syst. Ser. B 24 (2019) 1989–2015. https://doi.org/10.3934/dcdsb.2019026
[31] M. M. Meerschaert, F. Sabzikar, J. Chen, *Tempered fractional calculus*, J. Comput. Phys. 293 (2015) 14–28. https://doi.org/10.1016/j.jcp.2014.04.024
[32] S. T. M. Thabet, T. Abdeljawad, I. Kedim, M. I. Ayari, *A new weighted fractional operator with respect to another function via a new modified generalized Mittag-Leffler law*, Boundary Value Problems (2023) 2023:100. https://doi.org/10.1186/s13661-023-01790-7
[33] T. R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in the kernel*, Yokohama Mathematical Journal 19 (1971) 7–15.
[34] T. Abdeljawad, *Fractional operators with generalized Mittag-Leffler kernels and their iterated differintegrals*, Chaos 29 (2019) 1–10. https://doi.org/10.1063/1.5085726
[35] D. Baleanu, A. Fernandez, *On some new properties of fractional derivatives with Mittag-Leffler kernel*, Communications in Nonlinear Science and Numerical Simulation 59 (2018) 444–462. https://doi.org/10.1016/j.cnsns.2017.12.003