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Variable order R-L fractional calculus and its Applications

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Abstract

This paper presents a concise study of variable-order fractional calculus using the Riemann-Liouville approach. Specifically, we consider the Mittag-Leffler function with a single parameter as the order for both Riemann-Liouville fractional differentiation (FD) and fractional integration (FI). The study explores the impact of varying the parameter in the Mittag-Leffler (M-L) function and applies this variable-order fractional operator to polynomial functions of different degrees. For clarity and completeness, the behavior of the Mittag-Leffler-based Riemann-Liouville fractional calculus is examined both theoretically and graphically.

Keywords: Fractional Calculus, Gamma Function, Mittag - Leffler Function.
2010 MSC: 26A33, 33B15, 33E12.

1. Introduction

The "Fractional Calculus (FC)" is mainly reviewed as part of applied mathematics which handles explorations along with applications of differentiations and integrations of arbitrary order(see [1] and [2]). Thus, It is an prolongation of integer order Calculus that think of integrations as well derivatives of any complex or real order to unite also generalizes notions of integer-order differentials and n - fold integrals. Various forms of (FO)fractional operators have been introduced along time, like the Marchaud, Riemann–Liouville, Weyl, Grünwald – Letnikov, the Caputo, or Hadamard (FD)fractional derivatives(see [3], [4] and [5]).

An initial methodology is the "(R - L) Riemann – Liouville ", which is based on repeating the Classical Integral Operator n times and after it considers the Cauchy's formula where factorial is replaced by the Gamma function because of this the term called Fractional Integral of Non - Integer Order is outlined (see [2] and [3]). Therefore, By using this R - L operator, various (FD)fractional derivatives are defined which are stated above (see [4] and [5]).

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The term variable - order fractional operator (VO – FO) has now been focused and mathematically developed only in the current[6]. Foundation or base of of VO – FC is the various phenomenon or physical systems that occur in nature which depends upon the time domain and were especially used in fields like Viscoelastic Materials, Complex Media, Mechanics, Biomedical Engineering, Model – order reduction of lumped parameter systems and plasticity, Control theory and Optimization theory also in Time-dependent Transport processes in non – local elasticity[7]. VO-FC is a branch of calculus that offers tremendous opportunities to simulate integrative processes. The conceptualization of variable–order fractional calculus was mathematically conceptualized only in recent years by many mathematicians (see [8] and [9]). Samko and Ross were the first who propose the concept of variable-order integral and differential as well as some basic properties in 1993[10]. The research results of the variable-order fractional operators (VO – FO) after that investigation in the definitions of variable order fractional operators in different forms were summarized by Lorenzo and Hartley (see [11], [12] and [13]). We know that many physical systems change over time domain, even transitioning from a fractional order to another order, which gives main interest in fractional operators moving to their variable–order counterparts. Over the years, several proposals for fractional variable-order operators have appeared in the literature (see [14], [15] and [16]).

The Mittag - Leffler(M - L) function come to light in the solution of Fractional - Order Integral Equations or Fractional - Order Differential Equations, and principally in the L'evy flights, research of the fractional - generalization of the kinetic equation, in the exploration of Complex Systems, random walks and Super Diffusive Transport(see [17], [18], [19] and [20]).

Through out in these years we can see the evolution in (FC) Fractional Calculus. These functions has procured admiration and graveness on account on the basis of its massive applications in the field of engineering as well as in science[21]. Also in these few years, fascination in the Mittag - Leffler(M - L) function and functions like Mittag - Leffler(M - L) functions is significantly elevated among scientists as well as engineers due to their wide ability for explorations in several various applied queries(see [22] and [23]).

Fractional calculus (FC) and variable-order fractional calculus (VO-FC) have demonstrated remarkable versatility, with applications spanning diverse fields and domains. Advanced methods and frameworks have been developed to explore their properties and applications in detail. For instance, FC is employed in neural computing frameworks for forecasting non-standard nanofluid flow properties, incorporating novel physical parameters to model complex nanoscale behaviors[24]. Intricate fractal fractional-order simulations are used to analyze CO2 emissions from the energy sector, providing precise tools for environmental management[25]. In epidemiology, the piecewise modified ABC fractional order derivative has been applied to rotavirus infection models, offering deeper insights into disease dynamics. FC also proves valuable in analyzing media heterogeneity and network systems, enabling sophisticated studies of complex interactions[26]. Furthermore, singular fractional differential equations are solved using a modified Picard iterative approach and the psi-Caputo operator[27], while existence theorems for periodic boundary conditions in psi-fractional hybrid systems establish foundational principles for fractional dynamics[28]. These applications highlight the transformative potential of fractional calculus in modeling, simulation, and prediction across multidisciplinary domains.

This paper mainly divided into four parts, In the first part we have given an introduction towards variable order fractional calculus, Riemann Liouville approach of fractional calculus and Mittag - Leffler(M - L) function. The second part gives an extensive vision of theoretical development in the variable-order R - L Fractional derivative along with fractional integration respectively. The third part consists of conclusion part of our manuscript in detail and the last part we have references and bibliography for the same.

2. Theorotical Advancement:

2.1. Theorotical Advancement in Variable - Order (FD)Fractional derivative

Theorem 2.1. *The elementary representation of the variable-order R-L (FD)Fractional derivative* ${}_0^{\mathcal{RL}}D_1^{E_0(u)}g(u)$*, where g(u) = u and* $\alpha(u) = E_0(u)$ *is,*

$$ {}_0^{\mathcal{RL}}D_1^{E_0(u)}(u) = \frac{1}{\Gamma(3 - E_0(u))} $$

under the condition $u \in \mathbb{R}$

Proof: We Know that,

$$ E_0(u) = \frac{1}{1-u} \qquad ; u \in \mathbb{R}. $$

By definition of Variable Order R-L Fractional Derivative,

$$ {}_0^{\mathcal{RL}}D_1^{\alpha(u)}(u) = \frac{1}{\Gamma(1-\alpha(u))}\int_0^1 (u-\eta)^{(-\alpha(u))}g(\eta)d\eta $$

$$ \implies {}_0^{\mathcal{RL}}D_1^{E_0(u)}(u) = \frac{1}{\Gamma(1-E_0(u))}\int_0^1 (1-\eta)^{(-E_0(u))}(\eta)d\eta $$

$$ = \frac{1}{\Gamma(1-E_0(u))}\int_0^1 (1-\eta)^{[1-(E_0(u))-1]}(\eta)^{(2-1)}d\eta $$

$$ = \frac{1}{\Gamma(1-E_0(u))}\int_0^1 (1-\eta)^{[1-(E_0(u))]-1}(\eta)^{(2-1)}d\eta $$

By the definition of Beta function, we get,

$$ {}_0^{\mathcal{RL}}D_1^{E_0(u)}(u) = \frac{1}{\Gamma[1-E_0(u)]}B(1-(E_0(u)),2) $$

By using the property of Beta function, we get,

$$ {}_0^{\mathcal{RL}}D_1^{E_0(u)}(u) = \frac{1}{\Gamma[1-E_0(u)]}\frac{\Gamma[1-E_0(u)]\Gamma(2)}{\Gamma(1-E_0(u)+2)} $$

$$= \frac{\Gamma(2)}{\Gamma(1 - E_0(u) + 2)}$$

$$= \frac{1}{\Gamma(3 - E_0(u))}$$

$$\therefore {}_0^{\mathcal{RL}}D_1^{E_0(u)}(u) = \frac{1}{\Gamma(3 - E_0(u))} \qquad ; u \in \mathbb{R}$$

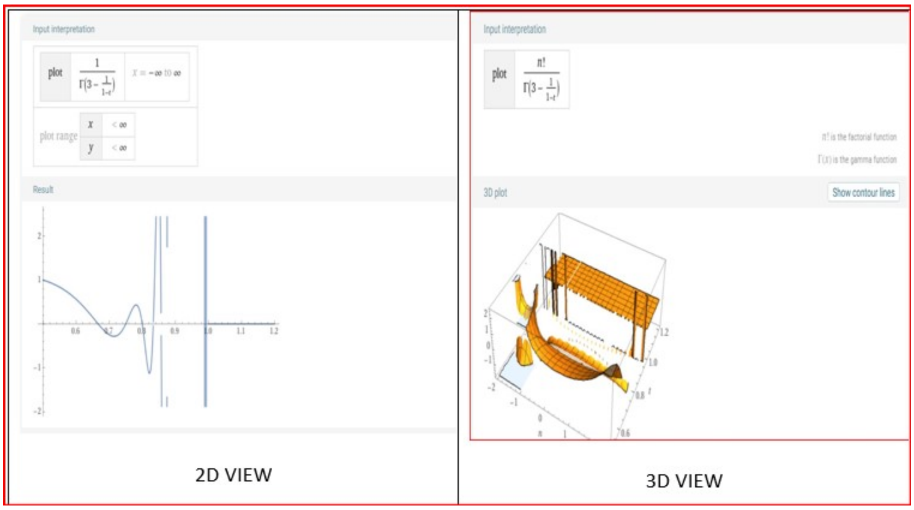

Figure 1:

Theorem 2.2. *The elementary representation of the variable-order R-L (FD)Fractional derivative* ${}_0^{\mathcal{RL}}D_1^{E_0(u)}h(u)$*, where* $h(u) = u^n$ *for* $\alpha(u) = E_0(u)$*, is given by*

$${}_0^{\mathcal{RL}}D_1^{E_0(u)}(u^n) = \frac{n!}{\Gamma(n + 2 - E_0(u))}$$

under the condition $u \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}$

Proof: We Know that,

$$E_0(u) = \frac{1}{1-u} \qquad ; u \in \mathbb{R}, E_0(u) \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}.$$

By the definition of variable-order R-L (FD)Fractional derivative,

$${}_0^{\mathcal{RL}}D_1^{\alpha(u)}(u^n) = \frac{1}{\Gamma(1 - \alpha(u))} \int_0^1 (u - \eta)^{(-\alpha(u))} h(\eta) d\eta$$

$$\implies {}_0^{\mathcal{RL}}D_1^{E_0(u)}(u^n) = \frac{1}{\Gamma(1 - E_0(u))} \int_0^1 (1 - \eta)^{(-E_0(u))} (\eta^n) d\eta$$

$$= \frac{1}{\Gamma(1 - E_0(u))} \int_0^1 (1 - \eta)^{[1-(E_0(u))-1]} (\eta)^{(n+1-1)} d\eta$$

$$= \frac{1}{\Gamma(1-E_0(u))} \int_0^1 (1-\eta)^{[1-(E_0(u))]-1} (\eta)^{(n+1)-1} d\eta$$

By the definition of Beta function, we get,

$$^{\mathcal{RL}}_{0}D_1^{E_0(u)}(u^n) = \frac{1}{\Gamma(1-E_0(u))} B(1-(E_0(u)),(n+1))$$

By using the property of Beta function, we get,

$$^{\mathcal{RL}}_{0}D_1^{E_0(u)}(u^n) = \frac{1}{\Gamma[1-E_0(u)]} \frac{\Gamma[1-E_0(u)]\Gamma(n+1)}{\Gamma[1-E_0(u)+(n+1)]}$$

$$= \frac{\Gamma[n+1]}{\Gamma[1-E_0(u)+(n+1)]}$$

$$= \frac{n!}{\Gamma(n+2-E_0(u))}$$

$$\therefore\, ^{\mathcal{RL}}_{0}D_1^{E_0(u)}(u^n) = \frac{n!}{\Gamma(n+2-E_0(u))}; \qquad u \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}$$

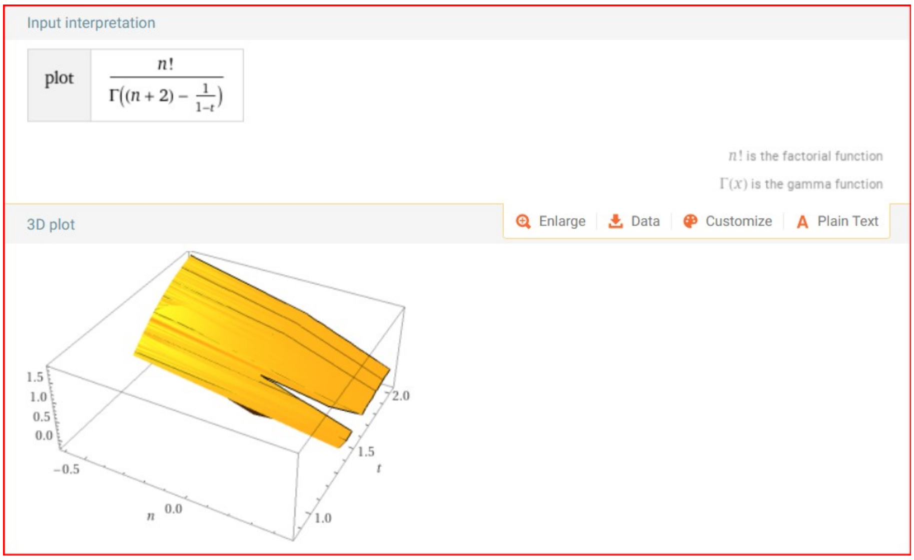

Figure 2:

3

Theorem 2.3. *The elementary representation of the variable-order R-L (FD)Fractional derivative* $^{\mathcal{RL}}_{0}D_1^{E_1(u)}(u^n)$*, where h(u)* $= u^n$ *for* $\alpha(u) = E_1(u)$*, is given by*

$$^{\mathcal{RL}}_{0}D_1^{E_1(u)}(u^n) = \frac{n!}{\Gamma(n+2-E_1(u))}$$

under the condition $u \in \mathbb{R}, E_1(u) \in (0,\infty), n \in \mathbb{N} \cup \{0\}$

Proof: We Know that,

$$E_1(u) = e^u \qquad ; u \in \mathbb{R}, E_1(u) \in (0,\infty), n \in \mathbb{N} \cup \{0\}.$$

By the definition of variable-order R-L (FD)Fractional derivative,

$$ {}_{0}^{\mathcal{RL}}D_1^{\alpha(u)}(u^n) = \frac{1}{\Gamma[1-\alpha(u)]} \int_0^1 (u-\eta)^{(-\alpha(u))} h(\eta) d\eta$$

$$\implies {}_{0}^{\mathcal{RL}}D_1^{E_1(u)}(u^n) = \frac{1}{\Gamma(1-E_1(u))} \int_0^1 (1-\eta)^{(-E_1(u))} (\eta^n) d\eta$$

$$= \frac{1}{\Gamma(1-E_1(u))} \int_0^1 (1-\eta)^{[1-(E_1(u))-1]} (\eta)^{(n+1)-1} d\eta$$

$$= \frac{1}{\Gamma(1-E_1(u))} \int_0^1 (1-\eta)^{[1-(E_1(u))]-1} (\eta)^{(n+1)-1} d\eta$$

By the definition of Beta function, we get,

$$ {}_{0}^{\mathcal{RL}}D_1^{E_1(u)}(u^n) = \frac{1}{\Gamma(1-E_1(u))} B(1-(E_1(u)),(n+1))$$

By using the property of Beta function, we get,

$$ {}_{0}^{\mathcal{RL}}D_1^{E_1(u)}(u^n) = \frac{1}{\Gamma(1-E_1(u)]} \frac{\Gamma[1-E_1(u)]\Gamma(n+1)}{\Gamma[1-E_1(u)+(n+1)]}$$

$$= \frac{\Gamma(n+1)}{\Gamma(1-E_1(u)+(n+1))}$$

$$= \frac{n!}{\Gamma((n+2)-E_1(u))}$$

$$\therefore {}_{0}^{\mathcal{RL}}D_1^{E_1(u)}(u^n) = \frac{n!}{\Gamma(n+2-E_1(u))}$$

where $u \in \mathbb{R}, E_1(u) \in (0,\infty), n \in \mathbb{N} \cup \{0\}$

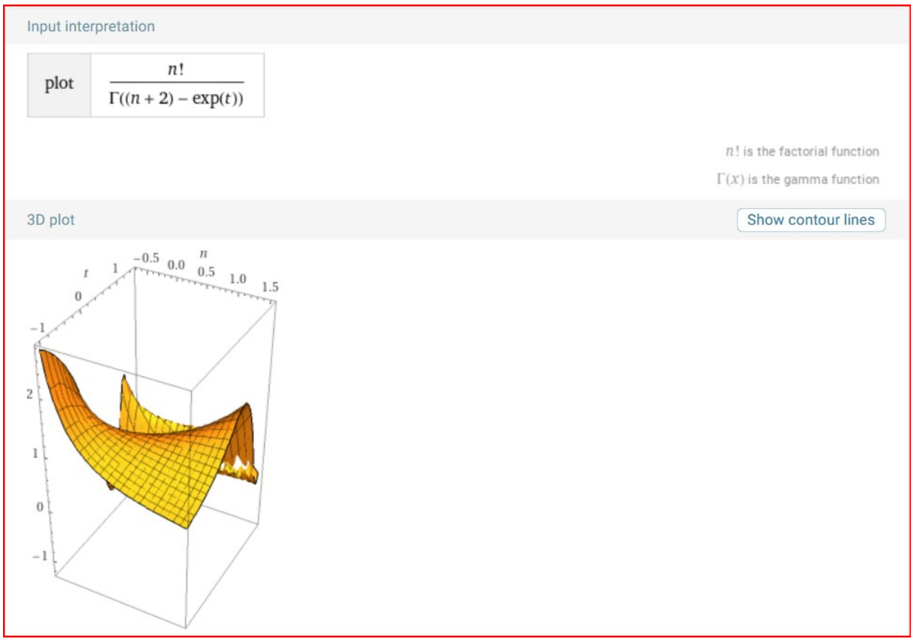

Figure 3:

Theorem 2.4. *The elementary representation of the variable-order R-L (FD)Fractional derivative* ${}^{\mathcal{RL}}_{0}D_{1}^{E_2(u)}(u^n)$, *where g(u)* $= u^n$ *for* $\alpha(u) = E_2(u)$, *is given by*

$$ {}^{\mathcal{RL}}_{0}D_{1}^{E(u)}(u^n) = \frac{n!}{\Gamma(n+2-E_2(u))} $$

under the condition where $u \in \mathbb{R}$ *such that* $E_2(u) \in [1,\infty), n \in \mathbb{N}\cup\{0\}$

Proof : We Know that,

$$ E_2(u) = \cosh(\sqrt{u}) $$

where $u \in \mathbb{R}$ such that $E_2(u) \in [1,\infty), n \in \mathbb{N}\cup\{0\}$.
By the definition of variable-order R-L (FD)Fractional derivative,

$$ {}^{\mathcal{RL}}_{0}D_{1}^{\alpha(u)}(u^n) = \frac{1}{\Gamma(1-\alpha(u))}\int_0^1 (u-\eta)^{(-\alpha(u))} g(\eta)d\eta $$

$$ \implies {}^{\mathcal{RL}}_{0}D_{1}^{E_2(u)}(u^n) = \frac{1}{\Gamma(1-E_2(u))}\int_0^1 (1-\eta)^{(-E_2(u))}(\eta^n)d\eta $$

$$ = \frac{1}{\Gamma(1-E_2(u))}\int_0^1 (1-\eta)^{[1-(E_2(u))-1]}(\eta)^{(n+1-1)}d\eta $$

$$ = \frac{1}{\Gamma(1-E_2(u))}\int_0^1 (1-\eta)^{[1-(E_2(u))]-1}(\eta)^{(n+1)-1}d\eta $$

By the definition of Beta function, we get,

$${}_{0}^{\mathcal{RL}}D_{1}^{E_2(u)}u^n) = \frac{1}{\Gamma(1-E_2(u))}B(1-(E_2(u)),(n+1))$$

By using the property of Beta function, we get,

$${}_{0}^{\mathcal{RL}}D_{1}^{E_2(u)}(^n) = \frac{1}{\Gamma[1-E_2(u)]}\frac{\Gamma[1-E_2(u)]\Gamma(n+1)}{\Gamma[1-E_2(u)+(n+1)]}$$

$$= \frac{\Gamma(n+1)}{\Gamma(1-E_2(u)+(n+1))}$$

$$= \frac{n!}{\Gamma((n+2)-E_2(u))}$$

$$\therefore {}_{0}^{\mathcal{RL}}D_{1}^{E_2(u)}(u^n) = \frac{n!}{\Gamma(n+2-E_2(u))}$$

where u$\in \mathbb{R}$ such that $E_2(u) \in [1,\infty), n \in \mathbb{N}\cup\{0\}$

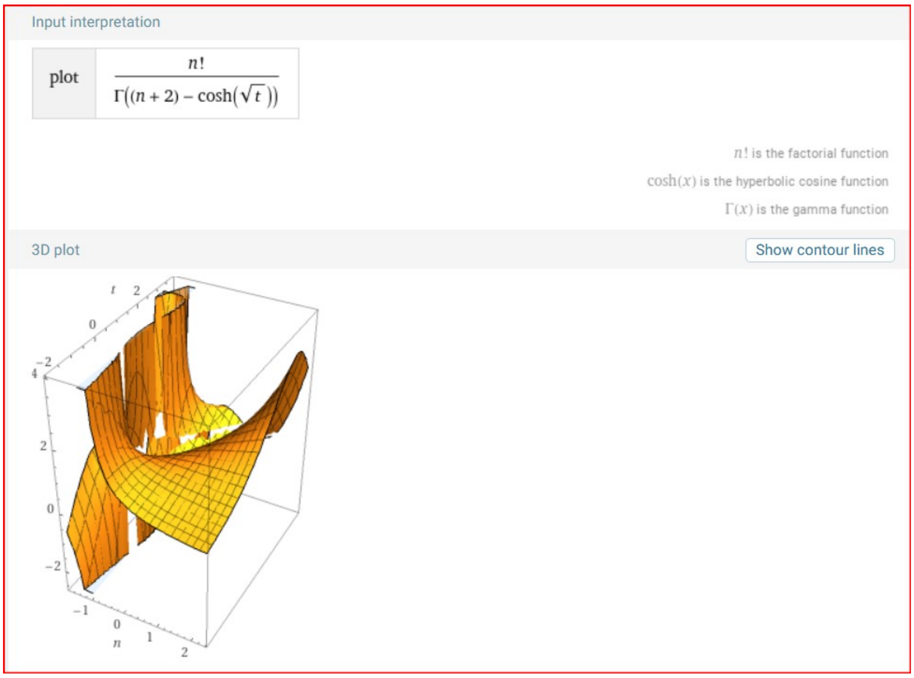

Figure 4:

Theorem 2.5. *The elementary representation of the variable-order R-L (FD)Fractional derivative* ${}_{0}^{\mathcal{RL}}D_{1}^{E_p(u)}(u^n)$, *h (u)* $= u^n$ *for* $\alpha(u) = E_p(u)$, *is given by*

$${}_{0}^{\mathcal{RL}}D_{1}^{E_p(u)}(u^n) = \frac{n!}{\Gamma(n+2-E_p(u))}$$

where $E_p(u)$ *is Mittag - Leffler function of one parameter with parameter* $\alpha = p$ *and* $u \in \mathbb{R}$ *such that* $E_p(u)$ *exist, u*$\in \mathbb{N}\cup\{0\}$

Proof: We Know that,

$$E_p(t) = \sum_{k=0}^{\infty} \frac{p^k}{\Gamma(pk+1)},$$

where $t \in \mathbb{R}$ such that $E_p(t)$ exist, $n \in \mathbb{N} \cup \{0\}$.
By the definition of variable-order R-L (FD)Fractional derivative,

$$ {}_0^{\mathcal{RL}}D_1^{\alpha(u)}(u^n) = \frac{1}{\Gamma(1-\alpha(u))} \int_0^1 (u-\eta)^{(-\alpha(u))} h(\eta) d\eta$$

$$\implies {}_0^{\mathcal{RL}}D_1^{E_p(u)}(u^n) = \frac{1}{\Gamma(1-E_p(u))} \int_0^1 (1-\eta)^{(-E_p(u))} (\eta^n) d\eta$$

$$= \frac{1}{\Gamma(1-E_p(u))} \int_0^1 (1-\eta)^{[1-(E_p(u))-1]} (\eta)^{(n+1-1)} d\eta$$

$$= \frac{1}{\Gamma(1-E_p(u))} \int_0^1 (1-\eta)^{[1-(E_p(u))]-1} (\eta)^{(n+1)-1} d\eta$$

By the definition of Beta function, we get,

$$ {}_0^{\mathcal{RL}}D_1^{E_p(u)}(u^n) = \frac{1}{\Gamma(1-E_p(u))} B((1-E_p(u)), n+1)$$

By using the property of Beta function, we get,

$$ {}_0^{\mathcal{RL}}D_1^{E_p(u)}(u^n) = \frac{1}{\Gamma[1-E_p(u)]} \frac{\Gamma[1-E_p(u)]\Gamma(n+1)}{\Gamma(1-E_p(u)+n+1)}$$

$$= \frac{\Gamma(n+1)}{\Gamma(1-E_p(u)+n+1)}$$

$$= \frac{n!}{\Gamma(n+2-E_p(u))}$$

$$\therefore {}_0^{\mathcal{RL}}D_1^{E_p(u)}(u^n) = \frac{n!}{\Gamma(n+2-E_p(u))}$$

where $u \in \mathbb{R}$ such that $E_p(u)$ exist and $n \in \mathbb{N} \cup \{0\}$

2.2. Theorotical Advancement in variable order fractional integration

Theorem 2.6. *The elementary representation of the variable-order R-L (FI)Fractional integration* ${}_0^{\mathcal{RL}}I_1^{E_0(u)}(u)$ *is given by*

$$ {}_0^{\mathcal{RL}}I_1^{E_0(u)}(u) = \frac{1}{\Gamma(E_0(u)+2)}$$

where $E_0(u)$ *is Mittag - Leffler function of one parameter with parameter* $\alpha = 0$, $u \in \mathbb{R}$

Proof: We Know that,

$$E_0(u) = \frac{1}{1-u} \qquad ; u \in \mathbb{R}.$$

By the definition of variable-order R-L (FI)Fractional integration,

$$\implies {}_0^{\mathcal{RL}}I_1^{E_0(u)}(u) = \frac{1}{\Gamma(E_0(u))}\int_0^1 (1-\eta)^{(E_0(u)-1)}(\eta)d\eta$$

$$= \frac{1}{\Gamma(E_0(u))}\int_0^1 (1-\eta)^{[(E_0(u))-1]}(\eta)^{(2-1)}d\eta$$

By the definition of Beta function, we get,

$${}_0^{\mathcal{RL}}I_1^{E_0(u)}(u) = \frac{1}{\Gamma(E_0(u))}B(E_0(u),2)$$

By using the property of Beta function, we get,

$${}_0^{\mathcal{RL}}D_1^{E_0(u)}(u) = \frac{1}{\Gamma[E_0(u)]}\frac{\Gamma[E_0(u)]\Gamma(2)}{\Gamma(E_0(u)+2)}$$

$$= \frac{\Gamma(2)}{\Gamma(E_0(u)+2)}$$

$$= \frac{1}{\Gamma(E_0(u)+2)}$$

$$\therefore {}_0^{\mathcal{RL}}I_1^{E_0(u)}(u) = \frac{1}{\Gamma(E_0(u)+2)} \qquad u \in \mathbb{R}, E_0(u) \in \mathbb{R}$$

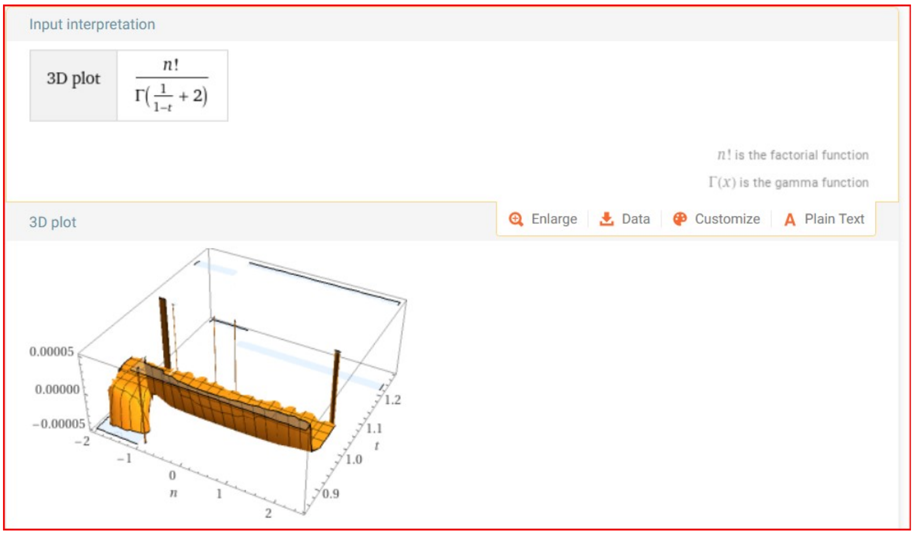

Figure 5:

Theorem 2.7. *The elementary representation of the variable-order R-L (FI)Fractional integration* ${}_0^{\mathcal{RL}}I_1^{E_0(u)}(u^n)$ *is given by*

$$ {}_0^{\mathcal{RL}}I_1^{E_0(u)}(u^n) = \frac{n!}{\Gamma(E_0(u)+n+1)} $$

where $E_0(t)$ *is Mittag - Leffler function of one parameter with parameter* $\alpha = 0$*, u*$\in \mathbb{R}$

Proof: We Know that,

$$ E_0(u) = \frac{1}{1-u} $$

where u$\in \mathbb{R}$, $E_0(u) \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}$.
By the definition of variable-order R-L (FI)Fractional integration,

$$ \implies {}_0^{\mathcal{RL}}I_1^{E_0(u)}(u^n) = \frac{1}{\Gamma(E_0(u))}\int_0^1 (1-\eta)^{[E_0(u)-1]}(\eta^n)d\eta $$

$$ = \frac{1}{\Gamma(E_0(u))}\int_0^1 (1-\eta)^{[E_0(u)-1]}(\eta)^{[n+1-1]}d\eta $$

By the definition of Beta function, we get,

$$ {}_0^{\mathcal{RL}}I_1^{E_0(u)}(u^n) = \frac{1}{\Gamma(E_0(u))}B(E_0(u), n+1) $$

By using the property of Beta function, we get,

$$ {}_0^{\mathcal{RL}}D_1^{E_0(u)}(u^n) = \frac{1}{\Gamma[E_0(u)]}\frac{\Gamma[E_0(u)]\Gamma(n+1)}{\Gamma(E_0(u)+n+1)} $$

$$ = \frac{\Gamma(n+1)}{\Gamma(E_0(u)+n+1)} $$

$$ = \frac{n!}{\Gamma(E_0(u)+n+1)} $$

$$ \therefore {}_0^{\mathcal{RL}}I_1^{E_0(u)}(u^n) = \frac{n!}{\Gamma(E_0(u)+n+1)} $$

where $u \in \mathbb{R}$, $E_0(u) \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}$

Theorem 2.8. *The elementary representation of the variable-order R-L (FI)Fractional integration* ${}_0^{\mathcal{RL}}I_1^{E_1(u)}(u^n)$ *is given by*

$$ {}_0^{\mathcal{RL}}I_1^{E_1(u)}(u^n) = \frac{n!}{\Gamma(E_1(u)+n+1)} $$

where $E_1(u)$ *is Mittag - Leffler function of one parameter with parameter* $\alpha = 1$*,* $u \in \mathbb{R}$*,* $E_1(u) \in (0,\infty)$*, n*$\in \mathbb{N} \cup \{0\}$

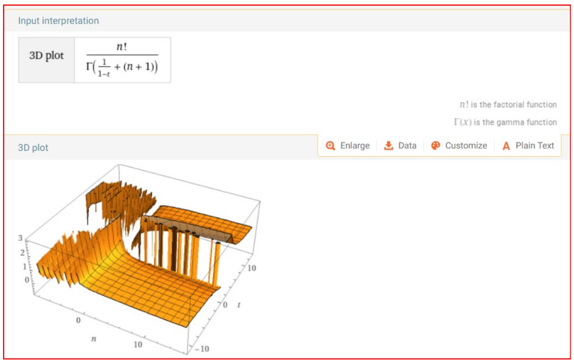

Figure 6:

Proof: We Know that,

$$E_1(u) = e^u$$

where $u \in \mathbb{R}$, $E_1(u) \in (0, \infty)$, $n \in \mathbb{N} \cup \{0\}$.
By the definition of variable-order R-L (FI)Fractional integration,

$$\implies {}_0^{\mathcal{RL}}I_1^{E_1(u)}(u^n) = \frac{1}{\Gamma[E_1(u)]}\int_0^1 (1-\eta)^{(E_1(u)-1)}(\eta^n)d\eta$$

$$= \frac{1}{\Gamma[E_1(u)]}\int_0^1 (1-\eta)^{[E_1(u)-1]}(\eta)^{[n+1-1]}d\eta$$

By the definition of Beta function, we get,

$${}_0^{\mathcal{RL}}I_1^{E_1(u)}(u^n) = \frac{1}{\Gamma[E_1(u)]}B(E_1(u), n+1)$$

By using the property of Beta function, we get,

$${}_0^{\mathcal{RL}}D_1^{E_1(u)}(t^n) = \frac{1}{\Gamma[E_1(u)]}\frac{\Gamma[E_1(u)]\Gamma(n+1)}{\Gamma[E_1(u)+n+1]}$$

$$= \frac{\Gamma(n+1)}{\Gamma[E_1(u)+n+1]}$$

$$= \frac{n!}{\Gamma[E_1(u)+n+1]}$$

$$\therefore {}_0^{\mathcal{RL}}I_1^{E_1(u)}(u^n) = \frac{n!}{\Gamma[E_1(u)+n+1]}$$

where $u \in \mathbb{R}$, $E_1(t) \in (0, \infty)$, $n \in \mathbb{N} \cup \{0\}$

Theorem 2.9. *The elementary representation of the variable-order R-L (FI)Fractional integration* ${}_{0}^{\mathcal{RL}}I_{1}^{E_2(u)}(u^n)$ *is given by*

$${}_{0}^{\mathcal{RL}}I_{1}^{E_2(u)}(u) = \frac{n!}{\Gamma(E_2(u)+n+1)}$$

where $E_2(u)$ *is Mittag - Leffler function of one parameter with parameter* $\alpha = 2$*,* $u \in \mathbb{R}$ *such that* $E_2(u) \in [1, \infty), n \in \mathbb{N} \cup \{0\}$

Proof: We Know that,

$$E_2(u) = \cosh(\sqrt{u})$$

where u$\in \mathbb{R}$ such that $E_2(u) \in [1, \infty), n \in \mathbb{N} \cup \{0\}$.

By the definition of variable-order R-L (FI)Fractional integration,

$$\implies {}_{0}^{\mathcal{RL}}I_{1}^{E_2(t)}(t^n) = \frac{1}{\Gamma[E_2(u)]}\int_0^1 (1-\eta)^{(E_2(u)-1)}(\eta^n)d\eta$$

$$= \frac{1}{\Gamma[E_2(u)]}\int_0^1 (1-\eta)^{[E_2(u)-1]}(\eta)^{[n+1-1]}d\eta$$

By the definition of Beta function, we get,

$${}_{0}^{\mathcal{RL}}I_{1}^{E_2(u)}(t^n) = \frac{1}{\Gamma[E_2(u)]}B(E_2(u), n+1)$$

By using the property of Beta function, we get,

$${}_{0}^{\mathcal{RL}}D_{1}^{E_2(u)}(u^n) = \frac{1}{\Gamma[E_2(u)]}\frac{\Gamma[E_2(u)]\Gamma(n+1)}{\Gamma(E_2(u)+n+1)}$$

$$= \frac{\Gamma[n+1]}{\Gamma(E_2(u)+n+1)}$$

$$= \frac{n!}{\Gamma(E_2(u)+n+1)}$$

$$\therefore {}_{0}^{\mathcal{RL}}I_{1}^{E_2(u)}(u^n) = \frac{n!}{\Gamma(E_2(u)+n+1)}$$

where u$\in \mathbb{R}$ such that $E_2(u) \in [1, \infty)$, n$\in \mathbb{N} \cup \{0\}$

Theorem 2.10. *The elementary representation of the variable-order R-L (FI)Fractional integration* ${}_{0}^{\mathcal{RL}}I_{1}^{E_p(u)}(u^n)$ *is given by*

$${}_{0}^{\mathcal{RL}}I_{1}^{E_p(u)}(u^n) = \frac{n!}{\Gamma(E_p(u)+n+1)}$$

where $E_p(u)$ *is Mittag - Leffler function of one parameter with parameter* $\alpha = p$ *and* $u \in \mathbb{R}$ *such that* $E_p(u)$ *exist,* $n \in \mathbb{N} \cup \{0\}$

Proof: We Know that,

$$E_p(u) = \sum_{k=0}^{\infty} \frac{p^k}{\Gamma(pk+1)},$$

where $u \in \mathbb{R}$ such that $E_p(u)$ exist, $n \in \mathbb{N} \cup \{0\}$.
By the definition of variable-order R-L (FI)Fractional integration,

$$\implies {}_{0}^{\mathcal{RL}}I_1^{E_p(u)}(u^n) = \frac{1}{\Gamma[E_p(u)]}\int_0^1 (1-\eta)^{(E_p(u)-1)}(\eta^n)d\eta$$

$$= \frac{1}{\Gamma[E_p(u)]}\int_0^1 (1-\eta)^{[E_p(u)-1]}(\eta)^{[n+1-1]}d\eta$$

By the definition of Beta function, we get,

$${}_{0}^{\mathcal{RL}}I_1^{E_p(u)}(u^n) = \frac{1}{\Gamma(E_p(u))}B(E_p(u),(n+1))$$

By using the property of Beta function, we get,

$${}_{0}^{\mathcal{RL}}D_1^{E_p(u)}(u^n) = \frac{1}{\Gamma[E_p(u)]}\frac{\Gamma[E_p(u)]\Gamma(n+1)}{\Gamma(E_p(u)+n+1)}$$

$$= \frac{\Gamma(n+1)}{\Gamma(E_p(u)+n+1)}$$

$$= \frac{n!}{\Gamma(E_p(u)+n+1)}$$

$$\therefore {}_{0}^{\mathcal{RL}}I_1^{E_p(u)}(u^n) = \frac{n!}{\Gamma(E_p(u)+n+1)}$$

where $u \in \mathbb{R}$ such that $E_p(u)$ exist, $n \in \mathbb{N} \cup \{0\}$

3. Conclusions:

This research provides a comprehensive analysis of Variable Order R-L Fractional Derivative and Integration for standard special functions, particularly the Mittag-Leffler function. The findings demonstrate the effectiveness of this approach in solving fractional differential equations with variable order and boundary conditions. Potential applications extend to partial fractional differential equations. Numerical and graphical analyses offer valuable insights into the impact of parameter variations on the Mittag-Leffler function and polynomial degree.

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