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First Step to Spectral Theory with Generalized M Derivative and Applications

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Abstract

In this article, several fundamental spectral results are established for the Sturm–Liouville problem with discrete boundary conditions involving the generalized M-derivative. The paper is organized into four sections. The first section provides a brief historical background of the topic. The second section presents essential definitions and foundational theorems. In the third section, we investigate the uniqueness theorem for the generalized M-derivative Sturm–Liouville boundary value problem on a finite interval and offer two distinct methods for representing the solution. The final section offers a comprehensive evaluation of the study, including a detailed visual analysis using graphical illustrations.

Keywords: Generalized M-derivative, Sturm-Liouville problem, Laplace transform, Direct problem.
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1. Introduction

Fractional analysis has been the focus of attention of many researchers since the past. Fractional analysis is the mathematical field that shows that the orders of integral and derivative operators are arbitrary numbers. Many mathematicians in this field have made many definitions using various notations [1]. Although fractional analysis applications have shown significant developments in various fields today, its basic mathematical history dates back to 300 years. The definition of derivative is made by Leibniz in 1695. Later, in the letter is written to L'Hospital Leibniz, the foundations of fractional analysis is laid on the question of whether the order of the derivative fractional [2]. As the usage areas of integral and derivative developed after Leibniz's definition, there is needed to further develop integral and derivative. The developed integral and derivative definitions are played a very important role in the solution of fractional order mathematical problem [3]. In addition, the application of various studies in classical analysis to fractional analysis has enabled obtaining precise results in solving many problems[4]. Studies have been done on

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analysis of fractional integral differential equations under Mittag-Leffler power law [5], [6]. In 1844, Boole is developed different methods for solving linear differential equations with constant coefficients and is used fractional calculation to solve the problem. Riemann is generalized the Taylor series and is given the definition of fractional integrals in 1847. Nowadays, Riemann-Liouville, Caputo and Grunwald-Letnikov fractional derivative definitions are used widely [7]. Additionally, Oliveira and Sousa are obtained the $\mathcal{M}-$derivative, which is a derivative in a new format containing the Mittag-Leffler function [8]. Spectral theory appears as an application area for various physical problems and many systems[9]. Therefore, boundary value problems involving differential equations have been studied in order to design assorted problems of engineering and mathematical physics. The solution to these problems are found by Charles François Sturm and Joseph Liouville in 1836 by the Sturm-Liouville theory. Today, the Sturm-Liouville theory continues to be the focus of attention of many researchers. This is because mathematical physics and quantum mechanics are the current problem. Various methods have been used to obtain the representation of the solution of the Sturm-Liouville problem [10]. $\mathcal{M}-$Laplace transform method is very important technique to find the solution or representation of differential equations [11].

2. Preliminaries

In this section, are given a little definitions and theorems necessary for our article.

Definition 2.1. [12] $f:[0,\infty)\to\mathfrak{R}$ function be defined. For $t>0$, $\beta>0$ and $\alpha\in(0,1)$ the truncated $\mathcal{M}-$series derivative of order α. of a function f is defined as

$$D^{\alpha}_{\mathcal{M}}f(t)=\lim_{\varepsilon\to 0}\frac{f\left(\Gamma(\gamma)t\mathcal{M}^{\beta,\gamma}_{p,r}(\varepsilon t^{-\alpha})\right)-f(t)}{\varepsilon}. \tag{2.1}$$

The following definitions have been defined already with $\mathcal{M}-$derivative and $\mathcal{M}-$series [11], [13], [10], [12], [14]. Here, we define these definitions using a different version of the Laplace transform of the generalized truncated $\mathcal{M}-$derivative and perform our operations accordingly.

Definition 2.2. Let $f:[a,\infty)\to\mathfrak{R}$, $\gamma,\beta>0$, $a\in\mathfrak{R}$ and $0<\alpha\leqslant 1$. The Laplace transform of the generalized truncated $\mathcal{M}-$derivative of the function f is

$$\mathcal{L}^{a}_{\alpha,\beta,\gamma}\{f(t)\}(s)=\int_a^\infty e^{-s\frac{\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right][t-a]^{\alpha}}{\alpha}}f(t)d_{\alpha}t \tag{2.2}$$

where $d_{\alpha}t=\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right](t-a)^{\alpha-1}dt.$

The Laplace transform of some functions is illustrated through the medium of the generalized truncated $\mathcal{M}-$derivative:

- $\mathcal{L}_{\alpha,\beta,\gamma}\left\{t^k\right\}(s)=\frac{\Gamma(1+\frac{k}{\alpha})\left(\alpha\left[\frac{a_1\cdots a_p}{c_1\cdots c_r}\cdot\frac{\Gamma(\gamma)}{\Gamma(\beta+\gamma)}\right]\right)^{\frac{k}{\alpha}}}{s^{\frac{k+\alpha}{\alpha}}}$

- $\mathcal{L}_{\alpha,\beta,\gamma}\left\{e^{n\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{t^\alpha}{\alpha}}\right\}(s)=\frac{1}{s-n} \qquad s>n.$
- $\mathcal{L}_{\alpha,\beta,\gamma}\left\{\sin\left(b\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{t^\alpha}{\alpha}\right)\right\}(s)=\frac{b}{b^2+s^2} \qquad s>0.$
- $\mathcal{L}_{\alpha,\beta,\gamma}\left\{\cos\left(b\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{t^\alpha}{\alpha}\right)\right\}(s)=\frac{s}{b^2+s^2} \qquad s>0.$

Definition 2.3. Let $h(t)$ and $g(t)$ are continuous functions and have exponential order, then the convolution of h and g for the generalized truncated $\mathcal{M}-$derivative is designated by

$$(h*g)(t)=\left[\int_a^t h(\tau)g(a+((t-a)^\alpha-(\tau-a)^\alpha)^{\frac{1}{\alpha}})d_\alpha\tau\right] \tag{2.3}$$

where $d_\alpha\tau=\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right](\tau-a)^{\alpha-1}d\tau.$

3. Main results

In this section, the uniqueness theorem for the non-integer order generalized $\mathcal{M}$-derivative regular Sturm-Liouville problem and the representation of the solution of this problem are given with two different methods. Let's represent the Sturm-Liouville operator L with the generalized $\mathcal{M}$-derivative as

$$L\equiv -D_{\mathcal{M}}^{2\alpha,\beta,\gamma}+q(x). \tag{3.1}$$

Here the function $0<\alpha\leqslant 1$, $[q(x)]$ is real and continuous in the spacing [a,b]. The aim of this section is to take into account the Sturm-Liouville problem with discrete boundary conditions:

$$Ly(x)=-D_{\mathcal{M}}^{2\alpha,\beta,\gamma}y(x)+[q(x)y(x)]=\lambda y(x). \tag{3.2}$$

$$y(a)\cos\alpha+D_{\mathcal{M}}^{\alpha,\beta,\gamma}y(a)\sin\alpha=0. \tag{3.3}$$

$$y(b)\cos\beta+D_{\mathcal{M}}^{\alpha,\beta,\gamma}y(b)\sin\beta=0. \tag{3.4}$$

If $\cot\beta=H$, $\cot\alpha=-h$ values are written for $x\in[0,\pi]$, the boundary conditions can be shown in the following form:

$$D_{\mathcal{M}}^{\alpha,\beta,\gamma}y(0)-hy(0)=0. \tag{3.5}$$

$$D_{\mathcal{M}}^{\alpha,\beta,\gamma}y(\pi)+Hy(\pi)=0. \tag{3.6}$$

It is worth noting that replacing the interval $[a,b]$ with the interval $[0,\pi]$ does not change the boundary conditions of 3.3 and 3.4. Therefore, it is assumed that $b=\pi$ and $a=0$. For any λ_n, the above boundary value problem has a non-trivial solution $[y(x,\lambda_n)]$. The initial condition corresponding to the $\phi(x,\lambda_n)$ solution of equation 3.2 is

$$[\phi(0,\lambda)]=1, D_{\mathcal{M}}^{\alpha,\beta,\gamma}[\phi(0,\lambda)]=h. \tag{3.7}$$

$$[\psi(0,\lambda)]=0, D_{\mathcal{M}}^{\alpha,\beta,\gamma}[\psi(0,\lambda)]=1. \tag{3.8}$$

Let the initial condition corresponding to the $\psi(x,\lambda)$ solution is given as 3.8.

Theorem 3.1 (Uniqueness Theorem). *Let presume the function* $[q(x)]$ *is continuous on the interval* $[a,b]$. *In this case, for each* ρ *the* $\phi(x,\lambda)$ *solution of equation 3.2 in the range* $a \leqslant x \leqslant b$ *is unique. It is shown as*

$$\phi(a,\lambda) = \sin\left(\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{\rho^\alpha}{\alpha}\right),$$
$$D_{\mathcal{M}}^{\alpha,\beta,\gamma}\phi(a,\lambda) = -\cos\left(\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{\rho^\alpha}{\alpha}\right). \quad (3.9)$$

Proof. Let choose the initial function in form

$$\phi_0(x,\lambda) = \sin\left(\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{\rho^\alpha}{\alpha}\right) - \left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\left(\frac{x^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right)\cos\left(\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{\rho^\alpha}{\alpha}\right). \quad (3.10)$$

This function is the solution of equation 3.2 that satisfies the conditions of 3.9. Let be

$$\phi_n(x,\lambda) = \phi_0(x,\lambda) + \int_a^x [q(t)-\lambda]\phi_{n-1}(t,\lambda)\left(\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right)d_\alpha t \quad (3.11)$$

for $n>0$. Since the function $q(x)$ is continuous, it is $|q(x)|<M$ in the interval $a \leqslant x \leqslant b$. Presume that there are $|\phi_0(x,\lambda)| \leqslant L$ and $|\lambda| \leqslant N$ for $a \leqslant x \leqslant b$. Therefore, for $n=1$ it becomes

$$\begin{aligned}|\phi_1(x,\lambda)-\phi_0(x,\lambda)| &= \int_a^x |q(t)-\lambda||\phi_0(t,\lambda)|\left|\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right|d_\alpha t\\ &\leqslant \int_a^x L[M+N]\left[\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right]d_\alpha t\\ &= \frac{L[M+N]}{2}\left[\frac{x^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right]^2. \end{aligned} \quad (3.12)$$

For the status $n \geqslant 2$, first of all find $\phi_{n-1}(x,\lambda)$ and subtract it from $\phi_n(x,\lambda)$

$$\phi_n(x,\lambda)-\phi_{n-1}(x,\lambda) = \int_a^x [q(t)-\lambda][\phi_{n-1}(t,\lambda)-\phi_{n-2}(t,\lambda)]\left[\frac{x^\alpha}{\alpha}-\frac{t^\alpha}{\alpha}\right]d_\alpha t \quad (3.13)$$

$$|\phi_n(x,\lambda)-\phi_{n-1}(x,\lambda)| \leqslant [M+N]\left[\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right]\int_a^x |\phi_{n-1}(t,\lambda)-\phi_{n-2}(t,\lambda)|d_\alpha t \quad (3.14)$$

is obtained. In case $n=2$ it becomes

$$\begin{aligned}|\phi_2(x,\lambda)-\phi_1(x,\lambda)| &\leqslant [M+N]\left[\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right]\int_a^x |\phi_1(t,\lambda)-\phi_0(t,\lambda)|d_\alpha t\\ &\leqslant \frac{L[M+N]^2}{2}\int_a^x\left[\frac{x^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right]^2 d_\alpha t\\ &= \frac{L[M+N]^2}{3!}\left[\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right]\left[\frac{x^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right]^3. \end{aligned} \quad (3.15)$$

If we generalized, to get

$$|\phi_n(x,\lambda)-\phi_{n-1}(x,\lambda)| \leqslant \frac{L[M+N]^n\left[\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right]^{n-1}\left[\frac{x^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right]^{n+1}}{(n+1)!}. \tag{3.16}$$

Therefore, the

$$\phi(x,\lambda)=\phi_0(x,\lambda)+\sum_{n=1}^{\infty}[\phi_n(x,\lambda)-\phi_{n-1}(x,\lambda)] \tag{3.17}$$

series converges smoothly to x and λ. Additionally,

$$D_{\mathcal{M}}^{\alpha,\beta,\gamma}\phi_n(x,\lambda)-D_{\mathcal{M}}^{\alpha,\beta,\gamma}\phi_{n-1}(x,\lambda)=\int_a^x[q(t)-\lambda][\phi_{n-1}(t,\lambda)-\phi_{n-2}(t,\lambda)]d_\alpha t \tag{3.18}$$

$$D_{\mathcal{M}}^{(2)\alpha,\beta,\gamma}\phi_n(x,\lambda)-D_{\mathcal{M}}^{(2)\alpha,\beta,\gamma}\phi_{n-1}(x,\lambda)=[q(t)-\lambda][\phi_{n-1}(t,\lambda)-\phi_{n-2}(t,\lambda)] \tag{3.19}$$

is obtained for $n \geqslant 2$. As a result, it becomes

$$\begin{aligned} D_{\mathcal{M}}^{(2)\alpha,\beta,\gamma}\phi(x,\lambda) &= \sum_{n=1}^{\infty}[D_{\mathcal{M}}^{(2)\alpha,\beta,\gamma}\phi_n(x,\lambda)-D_{\mathcal{M}}^{(2)\alpha,\beta,\gamma}\phi_{n-1}(x,\lambda)] \\ &= [D_{\mathcal{M}}^{(2)\alpha,\beta,\gamma}\phi_1(x,\lambda)-D_{\mathcal{M}}^{(2)\alpha,\beta,\gamma}\phi_0(x,\lambda)] \\ &\quad + \sum_{n=2}^{\infty}[D_{\mathcal{M}}^{(2)\alpha,\beta,\gamma}\phi_n(x,\lambda)-D_{\mathcal{M}}^{(2)\alpha,\beta,\gamma}\phi_{n-1}(x,\lambda)]. \end{aligned} \tag{3.20}$$

If equation 3.19 is substituted in equation 3.20,

$$\begin{aligned} D_{\mathcal{M}}^{(2)\alpha,\beta,\gamma}\phi(x,\lambda) &= [q(x)-\lambda]\left[\phi_0(x,\lambda)+\sum_{n=2}^{\infty}[\phi_{n-1}(x,\lambda)-\phi_{n-2}(x,\lambda)]\right] \\ &= [q(x)-\lambda]\phi(x,\lambda) \end{aligned} \tag{3.21}$$

is acquired. In this situation, it is seen that the $[\phi(x,\lambda)]$ function satisfies initial conditions and equation 3.2. Thus the proof is completed. Considering the given conditions, it is concluded that the uniqueness of ϕ is satisfied.
The Theorem, has a very significant role in finding the asymptotic formulas of eigenvalues and eigenfunctions, which have a very important place in spectral theory. In parallel with this result, let's obtain the representation of the solution for the problem considered using two methods.

Theorem 3.2. *Let $\lambda = s^2$. In this case it is in the form*

$$\begin{aligned} \phi(x,\lambda) &= \cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)+\frac{h}{s}\sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) \\ &\quad +\frac{1}{s}\int_0^x \sin\left[s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right]q(\tau)\phi(\tau,\lambda)d_\alpha\tau \end{aligned} \tag{3.22}$$

and

$$\psi(x,\lambda)=\frac{1}{s}\sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)+\frac{1}{s}\int_0^x\sin\left[s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right]q(\tau)\psi(\tau,\lambda)d_\alpha\tau. \quad (3.23)$$

Here $d_\alpha\tau=\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\tau^{\alpha-1}d\tau$ *is taken.*

Proof with Method of Variation of Parameters:

Let obtain the representation of the solution of equation 3.2 by using the method of change of constants shown in the previous section. Let the special solution y_p of equation 3.2 be

$$y_p(x)=v_1(x)\cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)+v_2(x)\sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right). \quad (3.24)$$

If we take the generalized $\mathcal{M}-$derivative of equation 3.24, get

$$D_{\mathcal{M}}^{\alpha,\beta,\gamma}y_p(x)=v_1(x)D_{\mathcal{M}}^{\alpha,\beta,\gamma}\left[\cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)\right]+\cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)D_{\mathcal{M}}^{\alpha,\beta,\gamma}v_1(x)+v_2(x)D_{\mathcal{M}}^{\alpha,\beta,\gamma}\left[\sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)\right]+\sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)D_{\mathcal{M}}^{\alpha,\beta,\gamma}v_2(x). \quad (3.25)$$

From here, equations

$$D_{\mathcal{M}}^{\alpha,\beta,\gamma}v_1(x)\cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)+D_{\mathcal{M}}^{\alpha,\beta,\gamma}v_2(x)\sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)=0 \quad (3.26)$$

$$D_{\mathcal{M}}^{\alpha,\beta,\gamma}v_1(x)\left[-s\sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)\right]+D_{\mathcal{M}}^{\alpha,\beta,\gamma}v_2(x)\left[s\cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)\right]=[q(x)-\lambda]y(x) \quad (3.27)$$

are achieved. The $\alpha,\beta,\gamma-$Wronskian of y_1 and y_2 is found to be

$$\begin{vmatrix}\cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) & \sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)\\ -s\sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) & s\cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)\end{vmatrix}=s. \quad (3.28)$$

Then, if equation

$$D_{\mathcal{M}}^{\alpha,\beta,\gamma} v_1(x) = \frac{\begin{vmatrix} 0 & \sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) \\ [q(x)-\lambda]y(x) & s\cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) \end{vmatrix}}{s}$$
$$= -\frac{1}{s}\sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)[q(x)-\lambda]y(x) \quad (3.29)$$

is integrated, it is found in the form

$$v_1(x) = -\frac{1}{s}\int_0^x \sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{\tau^\alpha}{\alpha}\right)\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]q(\tau)y(\tau)\tau^{\alpha-1}d\tau. \quad (3.30)$$

Similarly, if the equation

$$D_{\mathcal{M}}^{\alpha,\beta,\gamma} v_2(x) = \frac{\begin{vmatrix} \cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) & 0 \\ -s\sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) & [q(x)-\lambda]y(x) \end{vmatrix}}{s}$$
$$= \frac{1}{s}\cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)[q(x)-\lambda]y(x) \quad (3.31)$$

is integrated,

$$v_2(x) = \frac{1}{s}\int_0^x \cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{\tau^\alpha}{\alpha}\right)\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]q(\tau)y(\tau)\tau^{\alpha-1}d\tau \quad (3.32)$$

is found. If the values of $v_1(x)$ and $v_2(x)$ are substituted in equation 3.24, the special solution of $y_p(x)$ is obtained as

$$y_p(x) = \frac{1}{s}\int_0^x \sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right)q(\tau)y(\tau)d_\alpha\tau. \quad (3.33)$$

Here $d_\alpha\tau = \left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\tau^{\alpha-1}d\tau$ is taken. Therefore, the representation of the solution of equation 3.2 is in the form

$$y(x) = c_1y_1(x) + c_2y_2(x) + \frac{1}{W^{\alpha,\beta,\gamma}}\int_0^x [y_1(x)y_2(\tau) - y_2(x)y_1(\tau)]q(\tau)y(\tau)d_\alpha\tau. \quad (3.34)$$

Since
$y_1(x) = \cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)$, $y_2(x) = \sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)$

it is taken as

$$y(x) = c_1 \cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) + c_2 \sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) + \frac{1}{s}\int_0^x \sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right) q(\tau)y(\tau)d_\alpha\tau. \quad (3.35)$$

If the initial conditions 3.7 are used, the representation of solutions is obtained

$$\phi(x,\lambda) = \cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) + \frac{h}{s}\sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) + \frac{1}{s}\int_0^x \sin\left[s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right] q(\tau)\phi(\tau,\lambda)d_\alpha\tau. \quad (3.36)$$

Similarly, if the initial conditions of 3.8 are used, the other representation of the solution of equation 3.2 is

$$\psi(x,\lambda) = \frac{1}{s}\sin\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) + \frac{1}{s}\int_0^x \sin\left[s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right] q(\tau)\psi(\tau,\lambda)d_\alpha\tau. \quad (3.37)$$

Proof by means of Method of Laplace transform:
Let's obtain the representation of the solution of equation 3.2 with the help of the Laplace transform of the generalized truncated $\mathcal{M}-$derivative. First, let us consider the generalized truncated $\mathcal{M}-$derivative Sturm-Liouville problem with initial condition 3.7 as follows.

$$-D_{\mathcal{M}}^{(2)\alpha,\beta,\gamma}\phi(x) + q(x)\phi(x) = \lambda\phi(x). \quad (3.38)$$

If we apply the $\mathcal{L}_{\alpha,\beta,\gamma}$ transformation of both sides of the expression 3.38, it becomes

$$-\mathcal{L}_{\alpha,\beta,\gamma}[D_{\mathcal{M}}^{(2)\alpha,\beta,\gamma}\phi(x)] + \mathcal{L}_{\alpha,\beta,\gamma}[q(x)\phi(x)] = \mathcal{L}_{\alpha,\beta,\gamma}[\lambda\phi(x)]. \quad (3.39)$$

With the help of initial conditions 3.7, equality

$$\varphi_{\alpha,\beta,\gamma}(s) = \frac{s}{s^2+\lambda} + \frac{h}{s^2+\lambda} + \frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]x^\alpha}{\alpha}} q(x)\phi(x)d_\alpha x \quad (3.40)$$

is achieved. If the transformation $\mathcal{L}^{-1}_{\alpha,\beta,\gamma}$ is applied to both sides of equation 3.40,

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}[\varphi_{\alpha,\beta,\gamma}(s)] = \mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{s}{s^2+\lambda}\right] + \mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{h}{s^2+\lambda}\right] + \mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]x^\alpha}{\alpha}} q(x)\phi(x)d_\alpha x\right] \quad (3.41)$$

is obtained.

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}[\varphi_{\alpha,\beta,\gamma}(s)] = \phi(x,\lambda)$$

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{s}{s^2+\lambda}\right] = \cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)$$

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{h}{s^2+\lambda}\right] = \frac{h}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)$$

equations are written. The expression

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]x^\alpha}{\alpha}} q(x)\phi(x)d_\alpha x\right]$$

becomes

$$\frac{1}{\sqrt{\lambda}}\int_0^x \sin\left(\sqrt{\lambda}\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right)q(\tau)\phi(\tau,\lambda)d_\alpha\tau$$

by using the convolution feature. Thus, the representation of solution

$$\begin{aligned}\phi(x,\lambda) = {} & \cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) + \frac{h}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)\\ & + \frac{1}{\sqrt{\lambda}}\int_0^x \sin\left(\sqrt{\lambda}\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right)q(\tau)\phi(\tau,\lambda)d_\alpha\tau \end{aligned} \tag{3.42}$$

are accessed. Here $d_\alpha\tau = \left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\tau^{\alpha-1}d\tau$. If $\lambda = s^2$ is taken into account in equation 3.42, equation 3.36 obtained by the change of constants method is arrived.

Application 1.

$$-D^{2\alpha,\beta,\gamma}_{\mathcal{M}}\phi(x) + (x^2+1)\phi(x) = \lambda\phi(x) \tag{3.43}$$

Let obtain the representation of the solution of equation 3.43 with the aid of the Laplace transform of the generalized truncated $\mathcal{M}-$derivative.

If we apply $\mathcal{L}_{\alpha,\beta,\gamma}$ transformation of both sides of the expression 3.43, it becomes

$$-\mathcal{L}_{\alpha,\beta,\gamma}\left[D^{(2)\alpha,\beta,\gamma}_{\mathcal{M}}\phi(x)\right] + \mathcal{L}_{\alpha,\beta,\gamma}\left[(x^2+1)\phi(x)\right] = \mathcal{L}_{\alpha,\beta,\gamma}\left[\lambda\phi(x)\right]. \tag{3.44}$$

With the aid of initial conditions 3.7, equality

$$\varphi_{\alpha,\beta,\gamma}(s) = \frac{s}{s^2+\lambda} + \frac{h}{s^2+\lambda} + \frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]x^\alpha}{\alpha}}(x^2+1)\phi(x)d_\alpha x \tag{3.45}$$

is achieved. If the transformation $\mathcal{L}^{-1}_{\alpha,\beta,\gamma}$ is applied to both sides of equation 3.45,

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}[\varphi_{\alpha,\beta,\gamma}(s)] = \mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{s}{s^2+\lambda}\right] + \mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{h}{s^2+\lambda}\right] + \mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]x^\alpha}{\alpha}}(x^2+1)\phi(x)d_\alpha x\right] \quad (3.46)$$

is obtained.

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}[\varphi_{\alpha,\beta,\gamma}(s)] = \phi(x,\lambda)$$

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{s}{s^2+\lambda}\right] = \cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)$$

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{h}{s^2+\lambda}\right] = \frac{h}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)$$

equations are written. The expression

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]x^\alpha}{\alpha}}(x^2+1)\phi(x)d_\alpha x\right]$$

becomes

$$\frac{1}{\sqrt{\lambda}}\int_0^x \sin\left(\sqrt{\lambda}\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right)(\tau^2+1)\phi(\tau,\lambda)d_\alpha\tau$$

by using the convolution feature. Thus, the representation of solution

$$\phi(x,\lambda) = \cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) + \frac{h}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) + \frac{1}{\sqrt{\lambda}}\int_0^x \sin\left(\sqrt{\lambda}\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right)(\tau^2+1)\phi(\tau,\lambda)d_\alpha\tau \quad (3.47)$$

are accessed. Here $d_\alpha\tau = \left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\tau^{\alpha-1}d\tau$. $\lambda = s^2$ is taken.

Application 2.

$$-D^{2\alpha,\beta,\gamma}_{\mathcal{M}}\phi(x) + \sin(x)\phi(x) = \lambda\phi(x) \quad (3.48)$$

Let obtain the representation of the solution of equation 3.48 with the aid of the Laplace transform of the generalized truncated $\mathcal{M}$−derivative.

If we apply the $\mathcal{L}_{\alpha,\beta,\gamma}$ transformation of both sides of the expression 3.48, it becomes

$$-\mathcal{L}_{\alpha,\beta,\gamma}\left[D^{(2)\alpha,\beta,\gamma}_{\mathcal{M}}\phi(x)\right] + \mathcal{L}_{\alpha,\beta,\gamma}\left[\sin(x)\phi(x)\right] = \mathcal{L}_{\alpha,\beta,\gamma}\left[\lambda\phi(x)\right]. \quad (3.49)$$

With the help of initial conditions 3.7, equality

$$\varphi_{\alpha,\beta,\gamma}(s) = \frac{s}{s^2+\lambda} + \frac{h}{s^2+\lambda} + \frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]x^\alpha}{\alpha}} \sin(x)\phi(x)d_\alpha x \tag{3.50}$$

is achieved. If the transformation $\mathcal{L}^{-1}_{\alpha,\beta,\gamma}$ is applied to both sides of equation 3.50,

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}[\varphi_{\alpha,\beta,\gamma}(s)] = \mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{s}{s^2+\lambda}\right] + \mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{h}{s^2+\lambda}\right] + \mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]x^\alpha}{\alpha}} \sin(x)\phi(x)d_\alpha x\right] \tag{3.51}$$

is obtained.

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}[\varphi_{\alpha,\beta,\gamma}(s)] = \phi(x,\lambda)$$

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{s}{s^2+\lambda}\right] = \cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)$$

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{h}{s^2+\lambda}\right] = \frac{h}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right)$$

equations are written. The expression

$$\mathcal{L}^{-1}_{\alpha,\beta,\gamma}\left[\frac{1}{s^2+\lambda}\int_0^\infty e^{-s\frac{\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]x^\alpha}{\alpha}} \sin(x)\phi(x)d_\alpha x\right]$$

becomes

$$\frac{1}{\sqrt{\lambda}}\int_0^x \sin\left(\sqrt{\lambda}\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right)\sin(\tau)\phi(\tau,\lambda)d_\alpha\tau$$

by using the convolution feature. Thus, the representation of solution

$$\phi(x,\lambda) = \cos\left(s\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) + \frac{h}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\frac{x^\alpha}{\alpha}\right) + \frac{1}{\sqrt{\lambda}}\int_0^x \sin\left(\sqrt{\lambda}\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\left(\frac{x^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)\right)\sin(\tau)\phi(\tau,\lambda)d_\alpha\tau \tag{3.52}$$

are accessed. Here $d_\alpha\tau=\left[\frac{c_1...c_r}{a_1...a_p}\frac{\Gamma(\beta+\gamma)}{\Gamma(\gamma)}\right]\tau^{\alpha-1}d\tau$. $\lambda=s^2$ is taken.

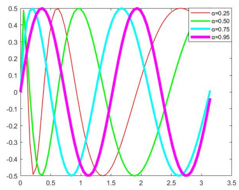

Figure 1: Evaluating the eigenfunction $\psi(x, \lambda)$

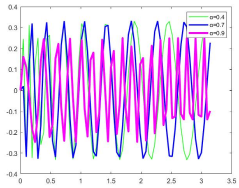

Figure 2: Representation of the solution of the eigenfunction $\psi(x, \lambda)$

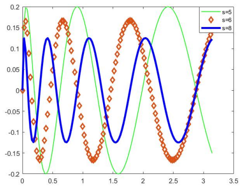

Figure 3: Graph of the solution representation of the eigenfunction $[\psi(x, \lambda)]$

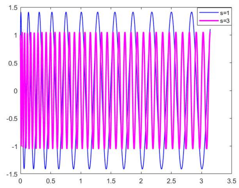

Figure 4: Graphical act of the solution representation of the eigenfunction $[\phi(x, \lambda)]$

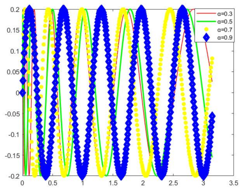

Figure 5: Graphical movement of the solution representation of the eigenfunction $[\psi(x, \lambda)]$

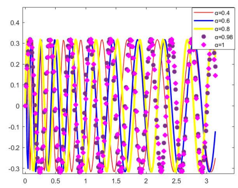

Figure 6: Graphical movement of the solution representation of the eigenfunction $[\psi(x, \lambda)]$

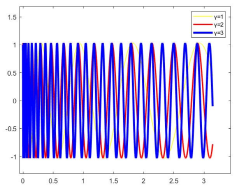

Figure 7: Visual of the eigenfunction $[\phi(x, \lambda)]$

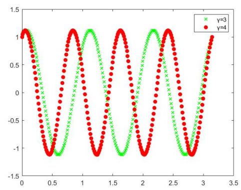

Figure 8: Visual of the eigenfunction $[\phi(x, \lambda)]$

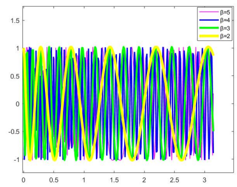

Figure 9: Behavior of the solution representation of the eigenfunction $[\phi(x, \lambda)]$

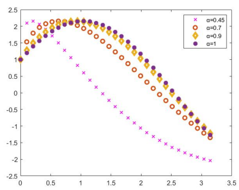

Figure 10: Act of the solution image of the eigenfunction $[\phi(x, \lambda)]$

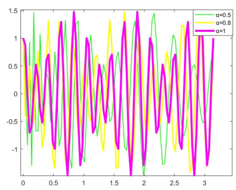

Figure 11: Visual of the eigenfunction for $[q(x) = x^2 + 1]$

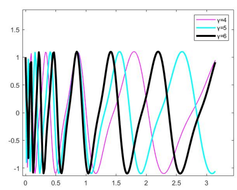

Figure 12: Visual of the eigenfunction for $[q(x) = \sin(x)]$

Figure 12, is obtained the representation of the eigenfunction 3.52 for the values s=2, β=1, γ=4, γ=5, γ=6 and α=0.3. Therefore, as a result of these examinations, it has been visually proven that when the values of $(a_1 \cdots a_p) = (c_1 \cdots c_r)$, α, β and γ are 1, overlap or very closely approximate the appearance of the classical solution. Moreover, the 3.2–3.4 solution of the generalized $\mathcal{M}$-derivative Sturm-Liouville problem is illustrated through the generalized $\mathcal{M}$-Laplace transform and the variation of generalized constants. The Sturm-Liouville problem considered, representation of the solution is obtained with the help of the generalized $\mathcal{M}$-Laplace transform by substituting (x^2+1) and $\sin(x)$ instead of the potential function $q(x)$.

4. Conclusions

In this section, the spectral construction of the Sturm-Liouville problem with discrete boundary conditions applied in the classical sense is examined through the generalized truncated $\mathcal{M}-$derivative. We exhibition an exhaustive for the unlike values of α, β, γ and s cantilevered by graphs. The visual behavior of the problems given the theory is examined. Figure 1, is shown the solution of $[\psi(x,\lambda)]$ by giving values β=2, s=2, γ=1, α=0.25, α=0.50, α=0.75 and α=0.95. In Figure 2, the behavior of the representation of the solution of the eigenfunction $[\psi(x,\lambda)]$ is examined by considering three different α values. Figure 3, is obtained the solution of the $[\psi(x,\lambda)]$ using the values γ=1, α=0.6, s=5, s=6, s=8 and β=1. Figure 4, is indicated the behavior of the solution representation of the eigenfunction for s=1, s=3, α=0.75, β=3 and γ=2 values. Figure 5, is shown the representation of equation $[\psi(x,\lambda)]$ for values of s=5, γ=2, β=1, α=0.3, α=0.5, α=0.7 and α=0.9. Figure 6, the representation of the solution is analyzed using the values $s=\pi$, β=2, γ=2, α=0.4, α=0.6, α=0.8, α=0.98 and α=1. Figure 7, the representation of the $[\phi(x,\lambda)]$ solution is analyzed for the values s=4, γ=1, β=2 and α=3/5. In Figure 8, the eigenfunction $[\phi(x,\lambda)]$ is examined by giving values s=2, γ=1, γ=3, γ=4 and β=1. Figure 9, the behavior of $[\phi(x,\lambda)]$ is observed by using the data of s=5, α=2/3, β=5, β=4, β=3, β=2, γ=1 and h=1. Figure 10, the eigenfunction $[\phi(x,\lambda)]$ is shown for the data $s=\pi/6$, β=2, γ=1, α=0.45, α=0.7, α=0.9, α=1 and h=1. Figure 11, is shown the eigenfunction 3.47 for s=2.5, β=2, γ=3, α=0.5, α=0.8 and α=1 values.
The generalized Sturm-Liouville problem is defined with respect to the generalized $\mathcal{M}$-derivative. This article is considered as the first step in finding spectral data. In the light of the data here, asymptotic formulas for eigenvalues, eigenfunctions and normalized numbers will be obtained in the future.

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