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Solvability of nonlinear coupled system of Urysohn-Volterra quadratic integral equations in generalized Banach Algebras

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Abstract

In this work, we investigate the solvability of a new class of nonlinear coupled systems of Urysohn-Volterra quadratic integral equations involving the generalized fractional kernel functions. By using the Leray-Schauder version of the fixed point theorem in the vectorial Banach algebra space, we prove the existence of solutions of the proposed system under suitable conditions. We investigate the stability analysis of the proposed system. Moreover, we establish some special examples and particular cases.

Keywords: Fixed point; Banach algebra, nonlinear integral equations, fractional integral, Leray-Schauder kind fixed point theorem.

1. Introduction

Integral equations innovate a very important field of functional analysis. This is due to the great importance of the integral equations because it is used in many applications, especially applications stemming from real life events such as nuclear energy [1], heat conduction [2], electromagnetic [3] and signal processing [4].

In addition, the fractional differential and integral equations are applicable in a wide range of other scientific subfields, such as mathematical modeling of emanations from energy sector [5], and the mathematical approach for diseases detection [6, 7, 8].

Operator equations create the basic tool of investigation conducted in the integral equations. In most cases, the proving of solvability of those equations of operators is done via applying fixed point approach. Many researchers established the fixed point theorems of sum and product of more than or equal three operators, (for examples, please see [9, 10, 11, 12, 13, 14, 15, 16].

In 2017, Hashem [17] established the solvability of the system of integral equations of Chandrasekhar kind

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$$\begin{aligned} u(t) &= F_1(t, u(t)) + G_1(t, v(t)) \int_0^t \frac{t}{t+s} U_1(s, v(s))ds, \quad t \in [0, a] \\ v(t) &= F_2(t, v(t)) + G_2(t, u(t)) \int_0^t \frac{t}{t+s} U_2(s, u(s))ds, \quad t \in [0, a], \end{aligned} \tag{1.1}$$

where, $(a > 0) \in \mathbb{R}_+$.The solvability results of the system (1.1) is obtained via Amar et al. [11] fixed point version for the block operator 2×2 matrix .

Fractional calculus is an essential and useful branch of mathematical analysis that investigated derivatives and integrals of fractional order. A long time ago, there are many definitions for fractional integrals operators, such as Riemann-Liouvilla, Hadamard, Katagampolg and Erdelyi-Kober fractional integral operators. Recently, in 2017, Almeida [18] proposed new definition of the fractional derivative and called this operator $\psi-$Caputo derivative. This new definition is more generalized then Riemann-Liouville, Hadamard, Erdely Kober and Caputo operators kinds.

In 2018, Darwish et al. [19] applied the approach of Darbo's fixed point to investigate the following Urysohn-Volterra integral equation

$$u(t) = f(t, u(t)) + g(t, u(t)) \int_0^t \frac{\psi'(s)(\psi(t) - \psi(s))^{p-1}}{\Gamma(p)} h(s, u(s))ds, \quad t \in [0, a], \tag{1.2}$$

where, $(a > 0) \in \mathbb{R}_+$ and $p \in (0, 1)$. The authors gave the existence of the solution of Equ.(1.2) under some certain conditions. In the same year, Nieto et al. [13] proposed some new versions of the fixed point theorems in algebra generalized Banach spaces. They established the type of Krasnosel'skki and Leray-Schauder fixed point for the product and sum of more than or equal two operators.

In 2019, Hahem et.al.[20], applying again the Amar et al. [11] fixed point approach to study the following system

$$\begin{aligned} u(t) &= \mathcal{F}_1(t, u(t)) + \mathcal{G}_1(t, v(t)) \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} \mathcal{S}_1(s, v(s))ds, \quad t \in [0, a], \\ v(t) &= \mathcal{F}_2(t, v(t)) + \mathcal{G}_2(t, u(t)) \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \mathcal{S}_2(s, u(s))ds, \quad t \in [0, a], \end{aligned} \tag{1.3}$$

where, $(a > 0) \in \mathbb{R}_+$ and $p, q \in (0, 1)$.

In 2020, Abdo [21] considered the existence results for at least one continuous solution for generalized fractional quadratic functional integral equation by using Schauder fixed point theorem.

Consider $I = [a, b]$. Let $C(I)$ be the Banach algebra of all continuous real-valued function on I with the supremum norm $\|u\|_\infty = \sup_{t \in I} |u(t)|$, $\forall\ u \in C(I)$ and pointwise product of functions. In this paper, we will apply the Nieto type fixed point theorem in Generalized vectorial algebra Banach space [13] to study the following nonlinear system of fractional integral equations:

$$u(t) = F_1(t,u(t),v(t)) + H_1(t,u(t),v(t)) \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} U_1(s,u(s),v(s))ds, \quad t \in [a,b]$$
$$v(t) = F_2(t,u(t),v(t)) + H_2(t,u(t),v(t)) \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\beta-1}}{\Gamma(\beta)} U_2(s,u(s),v(s))ds, \quad t \in [a,b], \tag{1.4}$$

where, $I = [a,b]$ such as $(b > a)$, $a \in \mathbb{R}_+ \cup \{0\}$, $b \in \mathbb{R}_+$ and $\alpha, \beta \in (0,1)$. The functions $F_1, F_2, H_1, H_2, U_1, U_2 : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous. The the function $\psi : I \to \mathbb{R}$ is a continuous, increasing, $\psi \in C^1(I,\mathbb{R})$ and $\psi'(t) \neq 0$ for all $t \in I$.

This article is organized as. Sect.2 is devoted to give some facts, basic results and definitions which will be used in the outcomes. In sect.3, we investigate the solvability of the system (1.4). In sect. 4, we establish the stability analysis of the system (1.4). Finally, we discuses some applications of the given results in Sect.5.

2. Preliminary

Throughout this paper, $\mathbb{R}^N_+$ will denote the set $\{u \in \mathbb{R}^N : a_i < 0 \quad \forall i = 1,2,...,N\}$. Let $u = (a_1, a_2, ..., a_N), v = (b_1, b_2, ..., b_N) \in \mathbb{R}^N$. Define the partial order $\preceq_N$ in $\mathbb{R}^N$ such that: $u \preceq_N v \Leftrightarrow a_i \leqslant b_i$ for all $i = 1,2,...,N$. Also , if $c \in \mathbb{R}$ then $u \leqslant c$ means $a_i \leqslant c$ for all $i = 1,2,...,N$. Furthermore, $|u| = (|a_1|,...,|a_N|)$ and $\max(u,v) = (\max\{a_1,b_1\},...,\max\{a_N,b_N\})$. Therefore, 0_N be the zero vector of $\mathbb{R}^N$. Next, we state the definition of generalized metric space, for more details , see [13, 22, 23, 24].

Definition 2.1 [25] Let $V \neq \varphi$ and $\rho : V \times V \to \mathbb{R}^N$, then ρ is said to be vector-valued-metric on V if for all $(u,v,w) \in V^3$ the following properties hold:

(1)$\rho(u,v) \succeq_N 0_N$;

(2) $\rho(u,v) = 0_N \Leftrightarrow u = v$;

(3) $\rho(u,v) = \rho(v,u)$;

(4)$\rho(u,v) \preceq_N \rho(u,w) + \rho(w,v)$.

(V,ρ) is called a generalized metric space and ρ is defined as

$$\rho(u,v) = \begin{pmatrix} \rho_1(u,v) \\ \rho_2(u,v) \\ \vdots \\ \rho_N(u,v) \end{pmatrix}.$$

Clearly, ρ is a generalized metric on V if and only if ρ_i are usual metrics, for all $i = 1,2,...,N$. We indicate that, the concepts of the sequences, cauchy sequences, convergence, closed and open subsets and completeness are the same like to those for usual metric spaces.

Definition 2.2 [13] Let V be a vectorial space over $\mathbb{R}$. A vector-valued generalized norm on V is a map $\|.\| : V \to \mathbb{R}^N_+$ such that for all $(u,v) \in V$ and $\kappa \in \mathbb{R}$ the following conditions hold:

1) $\|u\| \succeq_N 0_N$;

2) $\|u\| = 0_N \Leftrightarrow u = 0_N$;

3) $\|\kappa u\| = |\kappa|\|u\|$;

4) $\|u+v\| \preceq_N \|u\| + \|v\|$.

$(V, \|.\|)$ is said to be generalized normed space. If $(\rho(u, v) = \|u - v\|)$ is complete, then $(V, \|.\|)$ is said to be a generalized Banach space.

Definition 2.3 [13] Let $\mathcal{M}_{N\times N}(\mathbb{R}_+)$ be the family of all square matrices of size N which entries are positive real numbers. Suppose that, $L \in \mathcal{M}_{N\times N}(\mathbb{R}_+)$, then the spectral radius $\sigma(L)$ of L is defined as

$$\sigma(L) = \max\{|\lambda_j(L)| : i = 1, 2, ..., N\},$$

where $\lambda_j(L), i = 1, ..., N$ are the eigenvalues for L.

Definition 2.4 [13] The Generalized Banach algebra V is algebra that in the same time is also generalized Banach space such that for all $u, v \in V$, then

$$\|uv\| \preceq_N \|u\|\|v\|,$$

where

$$\|uv\| = \begin{pmatrix} \|uv\|_1 \\ \|uv\|_2 \\ \vdots \\ \|uv\|_N \end{pmatrix}$$

and

$$\|u\|\|v\| = \begin{pmatrix} \|u\|_1\|v\|_1 \\ \|u\|_2\|v\|_2 \\ \vdots \\ \|u\|_N\|v\|_N \end{pmatrix}.$$

Definition 2.5 [24] Let (V, ρ) be a generalized metric space. The map $\mathfrak{T} : V \to V$ is called contractive if $\exists\, L \in \mathcal{M}_{N\times N}(\mathbb{R}_+)$ such that $\sigma(L) < 1$ and

$$\rho(\mathfrak{T}u, \mathfrak{T}v) \preceq_N L\rho(u, v),$$

for all $(u, v) \in V^2$.

The following theorem is generalized Leray-Schauder version fixed point [13] which is the main tool to prove the results.

Theorem 2.1. *[13] Let V be a generalized Banach algebra, $0 \in \Omega$ be a bounded, convex and an open subset of V . suppose that $\mathcal{T}_1, \mathcal{T}_2 : V \to V$ and $\mathcal{T}_3 : \overline{\Omega} \to V$ are such that*

(1) $\mathcal{T}_1, \mathcal{T}_2$ are L_1, L_2 contractive respectively, where $L_1, L_2 \in \mathcal{M}_{N\times N}(\mathbb{R}_+)$, $L_1 = (\ell_{ij})_{1\leqslant i,j\leqslant N}$ and $L_2 = (\overline{\ell}_{ij})_{1\leqslant i,j\leqslant N}$;

(2) $\mathcal{T}_3$ is a completely continuous;

(3) $(\frac{I_V}{\mathcal{T}_1})^{-1}$ exists on $\mathcal{T}_3(\overline{\Omega})$, where $I_V(x) = x$ and $\frac{I_V}{\mathcal{T}_1} : V \to V$ begin defined as $(\frac{I_V}{\mathcal{T}_1})(x) = \frac{x}{\mathcal{T}_1 x}$;

(4) Let

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} \succeq_N \sup_{u\in\overline{\Omega}} \|\mathcal{T}_3 u\|$$

and

$$L^* = \begin{pmatrix} b_1\ell_{11} & \dots & b_N\ell_{1N} \\ \vdots & \ddots & \vdots \\ b_1\ell_{N1} & \dots & b_N\ell_{NN} \end{pmatrix} + \begin{pmatrix} \bar{\ell}_{11} & \dots & \bar{\ell}_{1N} \\ \vdots & \ddots & \vdots \\ \bar{\ell}_{N1} & \dots & \bar{\ell}_{NN} \end{pmatrix},$$

then $L^* \in \mathcal{M}_{N\times N}(\mathbb{R}_+)$ *and* $\sigma(L^*) < 1$.

Then either:

(I) there exists $u = \mathcal{T}_2(u) + \mathcal{T}_1(u)\mathcal{T}_3(u)$ *has a solution in* $\bar{\Omega}$*, or*
(II) there exist $u \in \bar{\Omega} \setminus \Omega$ *such that*

$$u = \lambda\mathcal{T}_2(\frac{u}{\lambda}) + \lambda\mathcal{T}_1(\frac{u}{\lambda})\mathcal{T}_3(u),$$

$\lambda \in (0, 1)$.

Next, we recall $\psi-$fractional integral operator, for more details, see [18].

Definition 2.6 Suppose that, $C^n(I, \mathbb{R}),\ n \in \mathbb{N}$, be the space of all $n-$times continuous and differentiable functions from I to $\mathbb{R}$. Let $\psi \in C^n(I, \mathbb{R})$, be an increasing functions such that $\psi'(t) \neq 0$ for all $t \in I$. Consider $u : I \to \mathbb{R}$ be integrable function. The $\psi-$Riemann-Liouville fractional integral of order $\alpha > 0,\ \alpha \in \mathbb{R}$ of the function u is defined as

$$J_{0^+}^{\alpha,\psi} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\zeta)(\psi(t) - \psi(\zeta))^{\alpha-1}\, u(\zeta)\, d\zeta\ ,$$

and the $\psi-$Riemann-Liouville fractional derivative of order $\alpha > 0,\ \alpha \in \mathbb{R}$ of the function u is defined as

$$D_{0^+}^{\alpha,\psi} u(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{1}{\psi'(t)}\frac{d}{dt})^n \int_0^t \psi'(\zeta)(\psi(t) - \psi(\zeta))^{n-\alpha-1}\, u(\zeta)\, d\zeta\ ,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integral part of α.

3. Existence Theory

The proving of the existence of solution of the proposed system (1.4) will be obtained under following assumptions:

(A1) there exits $q_{ij} \in \mathbb{R}_+$, $i, j = 1,2$ such that

$$|F_1(t,x_1,x_2) - F_1(t,y_1,y_2)| \leqslant \sum_{i=1}^{2} q_{1i}\, |x_i - y_i|$$

and

$$|F_2(t,x_1,x_2) - F_2(t,y_1,y_2)| \leqslant \sum_{i=1}^{2} q_{2i}\, |x_i - y_i|,$$

for all $t \in I$ and $(x_1,x_2,y_1,y_2) \in \mathbb{R}^4$;

(A2) there exits $p_{ij} \in \mathbb{R}_+$, $i,\ j = 1,2$ such that

$$|H_1(t,x_1,x_2) - H_1(t,y_1,y_2)| \leqslant \sum_{i=1}^{2} p_{1i}\, |x_i - y_i|$$

and

$$|H_2(t,x_1,x_2) - H_2(t,y_1,y_2)| \leqslant \sum_{i=1}^{2} p_{2i}\, |x_i - y_i|,$$

for all $t \in I$ and $(x_1,x_2,y_1,y_2) \in \mathbb{R}^4$;

(A3) $\sigma(P) < 1$ and $\sigma(Q) < 1$, where $Q = (q_{ij})_{1\leqslant i,j\leqslant 2}$ and $P = (p_{ij})_{1\leqslant i,j\leqslant 2}$;

(A4) $\exists\, m_i \in C(I)$, $i = 1,2$ and there exists $m_i^* \in \mathbb{R}_+, i = 1,2$ such that $m_i^* = \|m_i(t)\|_\infty, i = 1,2$ and

$$|U_1(t,u,v)| \leqslant m_1(t)$$

and

$$|U_2(t,u,v)| \leqslant m_2(t),$$

for all $t \in I$ and $(u,v) \in \mathbb{R}^2$;

(A5) $c^*[2+M] < 1$ where $c^* = \max\{p_{11}, p_{12}, p_{21}, p_{22}, q_{11}, q_{12}, q_{21}, q_{22}\}$, and $M = m_1^* \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)} + m_2^* \frac{(\psi(b)-\psi(a))^\beta}{\Gamma(\beta+1)}$.

Now, let $V = C(I) \times C(I)$.Define the generalized norm $\|.\| : V \to \mathbb{R}_+^2$ as

$$\|(u,v)\| = \begin{pmatrix} \|u\|_\infty \\ \|v\|_\infty \end{pmatrix}, \tag{3.1}$$

for all $(u,v) \in V$. Clearly $(V, \|.\|)$ is generalized Banach algebra. Let, $d : V \times V \to \mathbb{R}_+^2$ be the generalized metric space induced by norm which is defined as

$$d((u_1,v_1),(u_2,v_2)) = \begin{pmatrix} \|u_1 - u_2\|_\infty \\ \|v_1 - v_2\|_\infty \end{pmatrix}, \tag{3.2}$$

for all $(u_1,v_1),(u_2,v_2) \in V$

Define the operator $\mathcal{A}(u,v) = (\mathcal{A}_1(u,v), \mathcal{A}_2(u,v))$ where the superposition operator $\mathcal{A}_1, \mathcal{A}_2$ are defined as: $[\mathcal{A}_1(u,v)](t) = H_1(t,u(t),v(t))$ and $[\mathcal{A}_2(u,v)](t) = H_2(t,u(t),v(t))$. Also, let the operator $\mathcal{B}(u,v) = (\mathcal{B}_1(u,u), \mathcal{B}_2(u,v))$ where the superposition operator $\mathcal{B}_1, \mathcal{B}_2$ are defined as: $[\mathcal{B}_1(u,v)](t) = F_1(t,u(t),v(t))$ and $[\mathcal{B}_2(u,yv](t) = F_2(t,u(t),v(t))$. By the same argument, let $\mathcal{K}(x,y) = (\mathcal{K}_1(u,v), \mathcal{K}_2(u,v)$ where the superposition operators $\mathcal{C}_1, \mathcal{C}_2$ are defined as:

$$[\mathcal{K}_1(u,v)](t) = \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha}}{\Gamma(\alpha)} U_1(s,,u(s),v(s))ds$$

and

$$[\mathcal{K}_2(u,v)](t) = \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\beta}}{\Gamma(\beta)} U_2(s,u(s),v(s))ds.$$

Define $\mathbb{T} : V \to V$ as:

$$\mathbb{T}(u,v) = \mathcal{B}(u,v) + \mathcal{A}(u,v)\mathcal{K}(u,v). \tag{3.3}$$

Clearly, the system (1.4) has a solution if the operator $\mathbb{T}$ has a fixed point.

Lemma 3.1. *Suppose that the conditions A1-A3 hold. Then $\mathcal{A}, \mathcal{B}$ are contractive mappings.*

Proof. Let $(u_1,v_1),(u_2,v_2) \in V$, then we have

$$\begin{aligned} &|[\mathcal{A}_1(u_1,v_1)](t) - [\mathcal{A}_1(u_2,v_2)](t)| = |H_1(t,u_1(t),v_1(t)) - H_1(t,u_2(t),v_2(t))| \\ &\leqslant p_{11}|u_1(t)-u_2(t)| + p_{12}|v_1(t)-v_2(t)|. \end{aligned} \tag{3.4}$$

Therefore, we get

$$\|\mathcal{A}_1(u_1,v_1) - \mathcal{A}_1(u_2,v_2))\|_\infty \leqslant p_{11}\ \|u_1-u_2\|_\infty + p_{12}\ \|v_1-v_2\|_\infty. \tag{3.5}$$

Similarly, we have

$$\|\mathcal{A}_2(u_1,v_1) - \mathcal{A}_2(u_2,v_2))\|_\infty \leqslant p_{21}\ \|u_1-u_2\|_\infty + p_{22}\ \|v_1-v_2\|_\infty. \tag{3.6}$$

Hence, we obtain that

$$d(\mathcal{A}(u_1,v_1), \mathcal{A}(u_2,v_2)) \preceq_2 P \begin{pmatrix} \|u_1-u_2\|_\infty \\ \|v_1-v_2\|_\infty \end{pmatrix}. \tag{3.7}$$

Since $\sigma(P) < 1$ then $\mathcal{A}$ is contraction map.
Therefore, we get

$$\begin{aligned} &|[\mathcal{B}_1(u_1,v_1)](t) - [\mathcal{B}_1(u_2,v_2)](t)| = |F_1(t,u_1(t),v_1(t)) - F_1(t,u_2(t),v_2(t))| \\ &\leqslant q_{11}|u_1(t)-u_2(t)| + q_{12}|v_1(t)-v_2(t)|. \end{aligned} \tag{3.8}$$

Therefore, we get

$$\|\mathcal{B}_1(u_1,v_1) - \mathcal{B}_1(u_2,v_2))\|_\infty \leqslant q_{11}\ \|u_1-u_2\|_\infty + q_{12}\ \|v_1-v_2\|_\infty. \tag{3.9}$$

Similarly, we have

$$\|\mathcal{B}_2(u_1,v_1)-\mathcal{B}_2(u_2,v_2))\|_\infty \leqslant q_{21}\ \|u_1-u_2\|_\infty + q_{22}\ \|v_1-v_2\|_\infty. \tag{3.10}$$

Hence, we obtain that

$$d(\mathcal{B}(u_1,v_1),\mathcal{B}(u_2,v_2)) \preceq_2 Q \begin{pmatrix} \|u_1-u_2\|_\infty \\ \|v_1-v_2\|_\infty \end{pmatrix}. \tag{3.11}$$

Since $\sigma(Q)<1$ then $\mathcal{B}$ is contraction map.

□

Lemma 3.2. *Suppose that the conditions (A1)-(A5) hold. Then, there exists* K^* *such that, for every* $(u,v)\in V$ *solution of the following system*

$$\begin{aligned} u &= \lambda\ \mathcal{B}_1(\frac{u}{\lambda},\frac{v}{\lambda}) + \lambda\ \mathcal{A}_1(\frac{u}{\lambda},\frac{v}{\lambda})\mathcal{K}_1(u,v),\\ v &= \lambda\ \mathcal{B}_2(\frac{u}{\lambda},\frac{v}{\lambda}) + \lambda\ \mathcal{A}_2(\frac{u}{\lambda},\frac{v}{\lambda})\mathcal{K}_2(u,v), \end{aligned} \tag{3.12}$$

for some $\lambda\in(0,1)$*, we get* $\|u\|_\infty\leqslant K^*$ *and* $\|v\|_\infty\leqslant K^*$*.*

Proof. Let $(u,v)\in V$ be a solution of (3.12), then we have

$$\begin{aligned} \frac{u}{\lambda} &= \mathcal{B}_1(\frac{u}{\lambda},\frac{v}{\lambda}) + \mathcal{A}_1(\frac{u}{\lambda},\frac{v}{\lambda})\mathcal{K}_1(u,v),\\ \frac{v}{\lambda} &= \mathcal{B}_2(\frac{u}{\lambda},\frac{v}{\lambda}) + \mathcal{A}_2(\frac{u}{\lambda},\frac{v}{\lambda})\mathcal{K}_2(u,v), \end{aligned} \tag{3.13}$$

Let $\hat{u}=\frac{u}{\lambda}$ and $\hat{v}=\frac{v}{\lambda}$. We prove for an estimate of $\|\hat{u}\|_\infty$ and $\|\hat{v}\|_\infty$. The results obtained would be correct for $\|u\|_\infty$ and $\|v\|_\infty$. Then, we get

$$\begin{aligned} |\hat{u}(t)| &\leqslant |[\mathcal{B}_1(\hat{u},\hat{v})](t)| + |[\mathcal{A}_1(\hat{u},\hat{v})(t)\ |\ |\ [\mathcal{K}_1(\lambda\hat{u},\lambda\hat{v})](t)|\\ &\leqslant |F_1(t,\hat{u}(t),\hat{v}(t))-F_1(t,0,0)| + |F_1(t,0,0)|\\ &+ (|H_1(t,\hat{u}(t),\hat{v}(t))-H_1(t,0,0)| + |H_1(t,0,0)|)\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}|U_1(s,\lambda\ \hat{u}(s),\lambda\ \hat{v}(s))|ds\\ &\leqslant q_{11}|\hat{u}(t)| + q_{12}|\hat{v}(t)| + |F_1(t,0,0)|\\ &+ (p_{11}|\hat{u}(t)| + p_{12}|\hat{v}(t)| + |H_1(t,0,0)|)\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}|m_1(s)|ds. \end{aligned} \tag{3.14}$$

Let, $c^*=\max\{p_{11},p_{12},p_{21},p_{22},q_{11},q_{12},q_{21},q_{22}\}$, we get

$$\begin{aligned} |\hat{u}(t)| &\leqslant c^*(|\hat{u}(t)|+|\hat{v}(t)|)[1+m_1^*\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}ds]\\ &+ |F_1(t,0,0)| + |H_1(t,0,0)|\ m_1^*\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}ds. \end{aligned} \tag{3.15}$$

Similarly, we have

$$\begin{aligned}|\hat{v}(t)| \leqslant c^*(|\hat{u}(t)| + |\hat{v}(t)|)[1 + m_2^* \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\beta-1}}{\Gamma(\beta)} ds] \\ + |F_1(t,0,0)| + |H_1(t,0,0)|\ m_2^* \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\beta-1}}{\Gamma(\beta)} ds..\end{aligned} \tag{3.16}$$

Let, $f^* = \max\{\|F_1(t,0,0)\|_\infty, \|F_2(t,0,0)\|_\infty\}$ and $h^* = \max\{\|H_1(t,0,0)\|_\infty, \|H_2(t,0,0)\|_\infty\}$, then, we have

$$|\hat{u}(t)| \leqslant c^*(|\hat{u}(t)| + |\hat{v}(t)|)[1 + m_1^* \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}] + f^* + h^*\ m_1^* \frac{(\psi(t)-\psi(s))^\alpha}{\Gamma(\alpha+1)}. \tag{3.17}$$

Also, we have

$$|\hat{v}(t)| \leqslant c^*(|\hat{u}(t)| + |\hat{v}(t)|)[1 + m_2^* \frac{(\psi(b)-\psi(a))^\beta}{\Gamma(\beta+1)}] + f^* + h^*\ m_2^* \frac{(\psi(t)-\psi(s))^\beta}{\Gamma(\beta+1)}. \tag{3.18}$$

Let, $M = m_1^* \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)} + m_2^* \frac{(\psi(b)-\psi(a))^\beta}{\Gamma(\beta+1)}$, then by adding the last two equations we get,

$$|\hat{u}(t)| + |\hat{v}(t)| \leqslant c^*(|\hat{u}(t)| + |\hat{v}(t)|)[2 + M] + 2f^* + h^*\ M. \tag{3.19}$$

Let, $\mathbb{L} = c^*[2+M]$ and $\mathbb{K} = 2f^* + h^*M$. Then, if $\mathbb{L} < 1$, we get

$$|\hat{u}(t)| + |\hat{v}(t)| \leqslant \frac{\mathbb{K}}{1-\mathbb{L}}. \tag{3.20}$$

So, it is implies that

$$|u(t)| + |v(t)| \leqslant \frac{|\lambda|\ \mathbb{K}}{1-\mathbb{L}} \leqslant \frac{\mathbb{K}}{1-\mathbb{L}} = \mathbb{K}^*. \tag{3.21}$$

Thus, we get $\|u\|_\infty \leqslant \mathbb{K}^*$ and $\|v\|_\infty \leqslant \mathbb{K}^*$.

□

Lemma 3.3. *Suppose that the conditions (A1)-(A5) hold. The operator $\mathcal{K}$ is completely continuous.*

Proof. The proof is done in 3 steps.
Step 1. $\mathcal{K}$ is continuous. Let $\{(u_n, v_n)\}$ be a sequence in V such that $(u_n, v_n) \to (u, v) \in V$ as $n \to \infty$. Then we have $u_n \to u \in C(I)$ and $v_n \to v \in C(I)$ as $n \to \infty$. So, foe all $t \in I$, we get

$$
\begin{aligned}
&|[\mathcal{K}_1(u_n,v_n)](t)-[\mathcal{K}_1(u,v)](t)| \\
&=|\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\, U_1(s,u_n(s),v_n(s))ds-\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\, U_1(s,u(s),v(s))ds| \\
&\leqslant |\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha}}{\Gamma(\alpha)}\, |U_1(s,u_n(s),v_n(s))-U_1(s,u(s),v(s)|ds \\
&\leqslant \frac{(\psi(t)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\, \|U_1(t,x_n(t),v_n(t))-U_1(t,u(t),v(t)\|_\infty \\
&\leqslant \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\, \|U_1(t,x_n(t),v_n(t))-U_1(t,u(t),v(t)\|_\infty
\end{aligned} \tag{3.22}
$$

Thus, from the continuity of U_1, we have $\|\mathcal{K}_1(u_n,v_n)-\mathcal{K}_1(u,v)\|_\infty \to 0$ as $n\to\infty$. By doing the same steps, we get

$$
|[\mathcal{K}_2(u_n,v_n)](t)-[\mathcal{K}_2(u,v)](t)| \leqslant \frac{(\psi(b)-\psi(a))^{\beta}}{\Gamma(\beta+1)}\, \|U_2(t,x_n(t),v_n(t))-U_2(t,u(t),v(t)\|_\infty, \tag{3.23}
$$

$\|\mathcal{K}_2(u_n,v_n)-\mathcal{K}_2(u,v)\|_\infty \to 0$ as $n\to\infty$. Hence, we have $\|\mathcal{K}(u_n,v_n)-\mathcal{K}(u,v)\| \to 0$ as $n\to\infty$. Thus, $\mathcal{K}$ is continuous.

Step 2. Each bounded sets $\overline{\Omega}$ in V , $\mathcal{K}(\overline{\Omega})$ is uniformly bounded. Define, $\overline{\Omega}=\{(u,v)\in V : \|u\|_\infty \leqslant K^*, \|v\|_\infty \leqslant K^*\}$, where K^* is defined in lemma 3.2. Let $(u,v)\in\overline{\Omega}$, then we get

$$
\begin{aligned}
|[\mathcal{K}_1(u,v)](t)| &\leqslant \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\, |U_1(s,u(s),v(s)|)ds \\
&\leqslant \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha}}{\Gamma(\alpha)}\, |m_1(s))|ds \\
&\leqslant m_1^* \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} = b_1.
\end{aligned} \tag{3.24}
$$

similarly, we get

$$
\begin{aligned}
|[\mathcal{K}_2(u,v)](t)| &\leqslant \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\beta-1}}{\beta(\alpha)}\, |U_2(s,u(s),v(s)|)ds \\
&\leqslant \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\beta}}{\Gamma(\beta)}\, |m_2(s))|ds \\
&\leqslant m_2^* \frac{(\psi(b)-\psi(a))^{\beta}}{\Gamma(\beta+1)} = b_2.
\end{aligned} \tag{3.25}
$$

Thus, we have

$$
\|\mathcal{K}(u,v)\| \preceq_1 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{3.26}
$$

Hence, $\mathcal{K}(\overline{\Omega})$ is uniformly bounded in V.

Step 3. $\mathcal{K}$ is equi-continuous. Let $t_1,t_2\in I$ such that $t_1<t_2$, the we get

$$
\begin{aligned}
&|[\mathcal{K}_1(u,v)](t_1) - [\mathcal{K}_1(u,v)](t_2)| \\
&\leqslant |\int_a^{t_1} \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} U_1(s,u(s),v(s)ds - \int_a^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha}}{\Gamma(\alpha)} U_1(s,u(s),v(s)ds \\
&\leqslant m_1^*| \int_a^{t_1} \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} ds - \int_a^{t_1} \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} ds| \\
&\leqslant m_1^*|\frac{(\psi(t_1)-\psi(1))^{\alpha}}{\Gamma(\alpha+1)} - \frac{(\psi(t_2)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}|,
\end{aligned} \tag{3.27}
$$

therefore,

$$
\begin{aligned}
&|[\mathcal{K}_2(u,v)](t_1) - [\mathcal{K}_2(u,v)](t_2)| \\
&\leqslant |\int_a^{t_1} \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\beta-1}}{\Gamma(\beta)} U_2(s,u(s),v(s)ds - \int_a^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\beta}}{\Gamma(\beta)} U_2(s,u(s),v(s)ds \\
&\leqslant m_2^*| \int_a^{t_1} \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\beta-1}}{\Gamma(\beta)} ds - \int_a^{t_1} \frac{\psi'(s)(\psi(t_1)-\psi(s))^{\beta-1}}{\Gamma(\beta)} ds| \\
&\leqslant m_2^*|\frac{(\psi(t_1)-\psi(1))^{\beta}}{\Gamma(\beta+1)} - \frac{(\psi(t_2)-\psi(a))^{\beta}}{\Gamma(\beta+1)}|.
\end{aligned} \tag{3.28}
$$

From the uniform continuity of ψ , then when $t_1 \to t_2$, $|[\mathcal{K}_1(u,v)](t_1) - [\mathcal{K}_1(u,v)](t_2)| \to 0$ and $|[\mathcal{K}_2(u,v)](t_1) - [\mathcal{K}_2(u,v)](t_2)| \to 0$. Thus $\mathcal{K}$ is equi-continuous. So, it is implies that $\mathcal{K}$ is compact. Thus, $\mathcal{K} : \overline{\Omega} \to V$ is completely continuous. □

Theorem 3.4. *Let the conditions (A1)-(A5) hold, and assume that* $\sigma(W) < 1$ *where*

$$
W = \begin{pmatrix} b_1 p_{11} & b_2 p_{12} \\ b_1 p_{21} & b_2 p_{22} \end{pmatrix} + \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix},
$$

such that

$$
\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \succeq_2 \sup\{\|\mathcal{K}(u,v) : (u,v) \in \overline{\Omega}\},
$$

and $\Omega = \{(u,v) \in V : \|u\|_\infty < K^*, \|v\|_\infty < K^*\}$. *Then the system (1.4) has at least one solution in* V.

Proof. Since all the conditions of theorem 2.1 are satisfied. Then, the operator $\mathbb{T}$ has a fixed point $(u^*, v^*) \in V$. This fixed point is the solution of the system (1.4).

□

4. Asymptotically stability analysis

$(u,v) \in V$ is called an asymptotically stable solution of the system in the generalized Banach algebra V (1.4) if $\forall\ \varepsilon < 0$, $\exists\ T(T = T(\varepsilon) > 0$ such that for each $t \geqslant T(\varepsilon)$ and each (u^*, v^*) other solution of (1.4),

$$
\|(u,v) - (u^*, v^*)\| \preceq_2 \varepsilon.
$$

In the next theorem, we investigate the asymptotically stability analysis of the system (1.4).

Theorem 4.1. *Under the conditions (A1)-(A5), the system (1.4) is asymptotically stable.*

Proof. Let $(u, v) \in V$ be a solution of (1.4) and let (u^*, v^*) is other solution of (1.4), then we have

$$\begin{aligned}
|u(t) - u^*(t)| &\leqslant |F_1(t, u(t), v(t)) - F_1(t, u(t), v(t))| \\
&+ |H_1(t, u(t), v(t)) \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} U_1(s, u(s), v(s)) ds \\
&- H_1(t, u^*(t), v^*(t)) \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} U_1(s, u^*(s), v^*(s)) ds| \\
&\leqslant (q_{11} |u(t) - u^*(t)| + q_{12} |v(t) - v^*(t)|) \\
&+ |H_1(t, u(t), v(t)) \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} U_1(s, u(s), v(s)) ds \\
&- H_1(t, u(t), v(t)) \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} U_1(s, u^*(s), v^*(s)) ds| \\
&+ |H_1(t, u(t), v(t)) \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} U_1(s, u^*(s), v*(s)) ds \\
&- |H_1(t, u^*(t), v^*(t)) \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} U_1(s, u^*(s), v^*(s)) ds|
\end{aligned} \tag{4.1}$$

From the conditions (A1)-(A4), we get

$$\begin{aligned}
&|u(t) - u^*(t)| \leqslant c^*(|u(t) - u^*(t)| + |v(t) - v^*(t)|) \\
&+ |H_1(t, u(t), v(t))| \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} |U_1(s, u(s), v(s)) - U_1(s, u^*(s), v^*(s))| ds \\
&+ |H_1(t, u(t), v(t)) - H_1(t, u^*(t), v^*(t))| \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} |U_1(s, u^*(s), v*(s))| ds \\
&\leqslant c^*(|u(t) - u^*(t)| + |v(t) - v^*(t)|) \\
&+ 4\ m_1^*\ (c^*(\|u\|_\infty + \|v\|_\infty) + |H_1(t, 0, 0)|) \frac{(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha+1)} \\
&+ c^*\ m_1^*(|u(t) - u^*(t)| + |v(t) - v^*(t)|)) \frac{(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha+1)} \\
&\leqslant c^*(1 + m_1^*\ \frac{(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha+1)})(|u(t) - u^*(t)| + |v(t) - v^*(t)|) + 2\ m_1^*\ (2\ K^*\ c^* + h^*) \frac{(\psi(b) - \psi(a))^\alpha}{\Gamma(\alpha+1)}
\end{aligned} \tag{4.2}$$

Similarly, we have

$$\begin{aligned}|v(t)-v^*(t)| &\leqslant |F_2(t,u(t),v(t))-F_2(t,u(t),v(t))|\\&+|H_2(t,u(t),v(t))\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\beta-1}}{\beta(\alpha)}U_2(s,u(s),v(s))ds\\&-H_2(t,u^*(t),v^*(t))\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\beta-1}}{\Gamma(\beta)}U_2(s,u^*(s),v^*(s))ds|\\&\leqslant c^*(1+m_2^*\frac{(\psi(b)-\psi(a))^{\beta}}{\Gamma(\beta+1)})(|u(t)-u^*(t)|+|v(t)-v^*(t)|)\\&+2\,m_2^*\,(2\,K^*\,c^*+h^*)\frac{(\psi(b)-\psi(a))^{\beta}}{\Gamma(\beta+1)}\end{aligned} \tag{4.3}$$

Thus, we obtain that

$$|u(t)-u^*(t)|+|v(t)-v^*(t)|\leqslant c^*(2+M)(|u(t)-u^*(t)|+|v(t)-v^*(t)|)+2\,M\,(2\,K^*\,c^*+h^*) \tag{4.4}$$

According to the condition (A5), it follows that

$$|u(t)-u^*(t)|+|v(t)-v^*(t)|\leqslant \frac{2\,M\,(2\,K^*\,c^*+h^*)}{1-c^*(2+M)}=\varepsilon. \tag{4.5}$$

Hence, we have $|u(t)-u^*(t)|\leqslant\varepsilon$ and $|v(t)-v^*(t)|\leqslant\varepsilon$, and (1.4) is asymptotically stable. □

5. Applications and special cases

we present the following example, which indicate how the obtained results can be used to particular problems.

5.1. An Example

Consider the following system

$$\begin{aligned}u(t)&=\frac{|u(t)|+|v(t)|}{4+|u(t)|+|v(t)|}\\&+\frac{t^2+u(t)+v(t)}{te^{-t^2}+4}\int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\,\frac{(2s^2+e^{-s^2})(u(s)+v(s))}{20+u(s)+v(s)}ds,\quad t\in[0,1]\\v(t)&=\frac{t+\sin(|u(t)|+|v(t)|)}{4+t^2}\\&+\frac{1}{(t+2)^2}\arctan(|v(t)|+|u(t)|)\int_0^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\frac{2}{3}}}{\Gamma(\frac{2}{3})}\,\frac{\sin(u(s)+v(s))}{10+s^2+s^4}ds,\quad t\in[0,1].\end{aligned} \tag{5.1}$$

Let $\psi(t)=\frac{t^2+t}{2}$. Here, we get

$$F_1(t,u,v)=\frac{|u|+|v|}{4+|u|+|v|},$$

$$F_2(t,u,v) = \frac{t + \sin(|u| + |v|)}{4 + t^2},$$

$$H_1(t,u,v) = \frac{t^2 + u + v}{te^{-t^2} + 4},$$

$$H_2(t,u,v) = \frac{1}{(t+2)^2)} \arctan(|u| + |v|),$$

$$U_1(t,u,v) = \frac{(2t^2 + e^{-t^2})(u+v)}{20 + u + v},$$

$$U_2(t,u,v) = \frac{\sin(u+v)}{10 + t^2 + t^4}.$$

for all $t \in [0,1], u, v \in \mathbb{R}$. It is clear that all the maps F_1, F_2, H_1, H_2, U_1 and U_2 are continuous. Furthermore, we have:

$$|F_1(t,u_1,v_1) - F_1(t,u_2,v_2)| \leqslant \frac{1}{4}|u_1 - u_2| + \frac{1}{4}|v_1 - v_2|,$$

$$|F_2(t,u_1,v_1) - F_2(t,u_2,v_2)| \leqslant \frac{1}{4}|u_1 - u_2| + \frac{1}{4}|v_1 - v_2|,$$

$$|H_1(t,u_1,v_1) - H_1(t,u_2,v_2)| \leqslant \frac{1}{4}|u_1 - u_2| + \frac{1}{4}|v_1 - v_2|,$$

$$|H_2(t,u_1,v_1) - H_2(t,u_2,v_2)| \leqslant \frac{1}{4}|u_1 - u_2| + \frac{1}{4}|v_1 - v_2|,$$

for all $t \in [0,1]$ and $(u_1, u_2, v_1, v_2) \in \mathbb{R}$. It follows that:

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}, Q = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

Hence, we easily obtain $\sigma(P) = \frac{1}{2} < 1$, $\sigma(Q) = \frac{1}{2} < 1$ and $c^* = \frac{1}{4}$. Consequently, we have $m_1^* = m_2^* = \frac{1}{10}$. Therefore, we get

$$|U_1(t,u,v)| \leqslant m_1(t) = \frac{t}{10}$$

and

$$|U_2(t,u,v)| \leqslant m_2(t) = \frac{1}{10}.$$

Also, $b_1 = b_2 \cong \frac{1}{9}$, so, we get

$$W = \begin{pmatrix} b_1 p_{11} & b_2 p_{12} \\ b_1 p_{21} & b_2 p_{22} \end{pmatrix} + \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} \frac{5}{18} & \frac{5}{18} \\ \frac{5}{18} & \frac{5}{18} \end{pmatrix}.$$

It follows that: $\sigma(W) = \frac{5}{9} < 1$. Thus, the inequality $c^*(2 + M) \cong 2.56 < 1$ be verified. Hence, all the conditions from (A1)–(A5) are satisfied. From Theorem 3.4, we conclude that the system (5.1) has at least one solution and this solution is asymptotic stable.

5.2. Some special cases

The system (1.4) is very general fractional integral system. Consequently, the system include the following special cases:

(1) Putting $\alpha = \beta = 1$, then we have the following system:

$$\begin{aligned} u(t) &= F_1(t,u(t),v(t)) + H_1(t,u(t),v(t)) \int_a^t U_1(s,u(s),v(s))ds, \quad t \in [a,b] \\ v(t) &= F_2(t,u(t),v(t)) + H_2(t,u(t),v(t)) \int_a^t U_2(s,u(s),v(s))ds, \quad t \in [a,b]. \end{aligned} \tag{5.2}$$

(2) If $F_1(t,u,v) = f_1(t,u), F_1(t,u,v) = f_2(t,v), H_1(t,u,v) = g_1(t,v), H_2(t,u,v) = g_2(t,u), U_1(t,u,v) = S_1(t,v), U_2(t,u,v) = S_2(t,u)$ and $\psi(t) = t$, then we get the system (1.3).

(3) Letting $\psi(t) = t$, then we get the following system of integral equations of Riemann-Liouville kernel kind

$$\begin{aligned} u(t) &= F_1(t,u(t),v(t)) + H_1(t,u(t),v(t)) \int_a^t \frac{(t-s))^{\alpha-1}}{\Gamma(\alpha)} U_1(s,u(s),v(s))ds, \quad t \in [a,b] \\ v(t) &= F_2(t,u(t),v(t)) + H_2(t,u(t),v(t)) \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} U_2(s,u(s),v(s))ds, \quad t \in [a,b]. \end{aligned} \tag{5.3}$$

(4) Letting $\psi(t) = t^\gamma, \gamma \in \mathbb{R}_+$, then we get the following system of integral equations of Erdely-Kober kernel kind

$$\begin{aligned} u(t) &= F_1(t,u(t),v(t)) + H_1(t,u(t),v(t)) \int_a^t \frac{\gamma s^{\gamma-1}(t^\gamma - s^\gamma))^{\alpha-1}}{\Gamma(\alpha)} U_1(s,u(s),v(s))ds, \quad t \in [a,b] \\ v(t) &= F_2(t,u(t),v(t)) + H_2(t,u(t),v(t)) \int_a^t \frac{\gamma s^{\gamma-1}(t^\gamma - s^\gamma)^{\beta-1}}{\Gamma(\beta)} U_2(s,u(s),v(s))ds, \quad t \in [a,b]. \end{aligned} \tag{5.4}$$

(5)Letting $I = [1,e], \psi(t) = \ln(t), \gamma \in \mathbb{R}_+$, then we get the following system of weakly singular integral equations of Hadamard kernel kind

$$\begin{aligned} u(t) &= F_1(t,u(t),v(t)) + H_1(t,u(t),v(t)) \int_1^t \ln(\frac{t}{s})^{\alpha-1} \frac{1}{s\Gamma(\alpha)} U_1(s,u(s),v(s))ds, \quad t \in [1,e] \\ v(t) &= F_2(t,u(t),v(t)) + H_2(t,u(t),v(t)) \int_1^t \ln(\frac{t}{s})^{\beta-1} \frac{1}{s\Gamma(\beta)} U_2(s,u(s),v(s))ds, \quad t \in [1,e]. \end{aligned} \tag{5.5}$$

Finally we can state the following results for the above special case

Theorem 5.1. *Under the conditions (A1)-(A5), the the following system of integral equations of Riemann-Liouville type has at least one asymptotically stable solution in* V.

$$\begin{aligned} u(t) &= F_1(t,u(t),v(t)) + H_1(t,u(t),v(t)) \int_a^t \frac{(t-s))^{\alpha-1}}{\Gamma(\alpha)} U_1(s,u(s),v(s))ds, \quad t \in [a,b] \\ v(t) &= F_2(t,u(t),v(t)) + H_2(t,u(t),v(t)) \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} U_2(s,u(s),v(s))ds, \quad t \in [a,b]. \end{aligned} \tag{5.6}$$

Theorem 5.2. *the following system of integral equations of Erdely Kober type has at least one asymptotically stable solution in* V,

$$u(t) = F_1(t,u(t),v(t)) + H_1(t,u(t),v(t)) \int_a^t \frac{\gamma s^{\gamma-1}(t^\gamma - s^\gamma))^{\alpha-1}}{\Gamma(\alpha)} U_1(s,u(s),v(s))ds, \quad t \in [a,b]$$
$$v(t) = F_2(t,u(t),v(t)) + H_2(t,u(t),v(t)) \int_a^t \frac{\gamma s^{\gamma-1}(t^\gamma - s^\gamma)^{\beta-1}}{\Gamma(\beta)} U_2(s,u(s),v(s))ds, \quad t \in [a,b]. \tag{5.7}$$

Theorem 5.3. *Under the conditions (A1)-(A5), the following system of weakly singular integral equations of Hadamard type has at least one asymptotically stable solution in* V,

$$u(t) = F_1(t,u(t),v(t)) + H_1(t,u(t),v(t)) \int_1^t \ln(\frac{t}{s})^{\alpha-1} \frac{1}{s\Gamma(\alpha)} U_1(s,u(s),v(s))ds, \quad t \in [1,e]$$
$$v(t) = F_2(t,u(t),v(t)) + H_2(t,u(t),v(t)) \int_1^t \ln(\frac{t}{s})^{\beta-1} \frac{1}{s\Gamma(\beta)} U_2(s,u(s),v(s))ds, \quad t \in [1,e]. \tag{5.8}$$

6. Conclusion

In this article, a nonlinear system of integral equation with $\psi-$kernels is considered in generalized Banach algebras. We investigate the solvability of the proposed system via generalized Leray-Schauder fixed point approach. The stability analysis of the proposed system was studied. The reported results in this paper are recent and significantly contribute to the existing literature on the subject. It was concluded that the proposed system is very general and involves many special cases, we gave some of those special cases and illustrative example.

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