SABA Publishing

Journal of Fractional Calculus and Nonlinear Systems
jfcns.sabapub.com
ISSN : 2709-9547

J Frac Calc & Nonlinear Sys
(2024)5(1) : 12-31
doi:10.48185/jfcns.v5i1.1118

Linear $\alpha-$Differential Equations

MOHAMMED SHEHATA*

Department of Basic Science, Bilbeis Higher Institute For Engineering, Sharqia, Egypt

• *Received: 15 April 2024* • *Accepted: 10 June 2024* • *Published Online: 23 June 2024*

Abstract

By $\alpha-$differential equations, we mean that the branch studies differential equations containing fractional or real order derivatives. In [1], Shehata has overcome the problem of multiple previous definitions of fractional calculus by putting an accurate definition of the $\alpha-$ fractional calculus using the normal way. He concluded from this definition that the α fractional calculus is a complex-valued function that depends on the principal root of the real number. As an extension of the study of fractional calculus and its importance in applications, in this paper, we study differential equations that contain fractional differentials based on the Shehata definition. We define and study the so-called linear $\alpha-$ differential equations of the first extension, higher extension, and system of the first extension. We give the closed formula for each case. To illustrate our result, we give some numerical examples of fractional differential equations and their solutions.

Keywords: Fractional differential equations, α-calculus, fractional calculus.
2010 MSC: 34A08, 26A33, 34k37.

1. Introduction

$\alpha-$(fractional) differential equation is a generalization of ordinary differential equations to arbitrary non integer orders. research has shown that physical systems can be represented more accurately through fractional derivative formulation, it is so important that in such a short span of time, it has touched all the classical field of mathematics, chemistry, engineering, physics, economics etc. [2]-[9].

There have been many monographs and books available on this field. Most recent and up to-date developments on fractional calculus with applications involving many different potentially useful operators of fractional calculus was given by many (see for example [10]-[13]).

The concept of fractional calculus(α derivatives and α integral) is not new, the idea of fractional calculus has been suggested by L'Hospital at the end of the seventeenth century. Since then, a strong theoretical foundations about fractional calculus were carried out in the last and present centuries. Many researchers used an integral form for fractional derivative definition. The most popular definitions of fractional derivative are Riemann-Liouville and Caputo definitions.

*Corresponding author: mashehata_math@yahoo.com

Definition 1.1. (Riemann-Liouville fractional derivative 1847). Let $f(.)$ be a continuous function in $[a,b]$, $0<\alpha\leqslant 1$.

The left Riemann-Liouville fractional derivative of order α is given by

$$_aD_t^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\left(\int_a^t(t-\tau)^{-\alpha}f(\tau)d\tau\right),\quad t\in[a,b].$$

The right Riemann-Liouville fractional derivative of order α is given by

$$_tD_b^{\alpha}f(t)=-\frac{1}{\Gamma(\alpha)}\frac{d}{dt}\left(\int_t^b(\tau-t)^{-\alpha}f(\tau)d\tau\right),\quad t\in[a,b].$$

Definition 1.2. (Caputo's fractional derivatives 1967). Let $f(.)$ be a continuous function in $[a,b]$, $0<\alpha\leqslant 1$. The left Caputo fractional derivative of order α is given by

$$_a^CD_t^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t(t-\tau)^{-\alpha}\frac{d}{dt}f(\tau)d\tau,\quad t\in[a,b].$$

and the right Caputo fractional derivative derivative of order α is given by

$$_t^CD_b^{\alpha}f(t)=-\frac{1}{\Gamma(1-\alpha)}\int_t^b(\tau-t)^{-\alpha}\frac{d}{dt}f(\tau)d\tau,\quad t\in[a,b].$$

But this definitions gives rise to some problems. For example if one makes $f(t)=C=constant$ in Riemann-Liouville formula, one finds that its α^{th} derivative is $\frac{Ct^{\alpha}}{\Gamma(1-\alpha)}$, that is to say, is different from zero. This is because they depended on the integral on the definition of the derivative.

A new fractional $\alpha-$ calculus definition based on classical definition is given in our paper [1]. In [1], we proved that $\alpha-$ derivative and $\alpha-$ integral are complex valued functions depends on the principal root.

This paper is devoted to some questions in $\alpha-$ calculus, that is, the theory of differential and integral operators of non-integer order, and in particular to so-called $\alpha-$ differential equations. Through this paper $\alpha\in(0,1]$.

2. α- Differentiable function

Here, some basic definitions and properties of the α calculus theories which can be found in [1].

Definition 2.1. (principal exponential of real number) For a real number t, we denote by $(t)^{\alpha}$ to the $\alpha-$ principal exponential of t which is defined by

$$\begin{aligned}(t)^{\alpha}:=|t|^{\alpha}.&\begin{cases}1 & t\geqslant 0\\ e^{i\pi\alpha} & t<0\end{cases}\\ =|t|^{\alpha}.&\begin{cases}1 & t\geqslant 0\\ \cos\alpha\pi+i\sin\alpha\pi & t<0\end{cases}\end{aligned}$$

For examples

$$(4)^{\frac{1}{2}}=2, \quad (-4)^{\frac{1}{2}}=2i, \quad (8)^{\frac{1}{3}}=2, \quad (-8)^{\frac{1}{3}}=1+\sqrt{3}i$$

From the above definition, we can define $e^{t^{\alpha}}$ by

$$\begin{aligned}
e^{t^{\alpha}} &= \begin{cases} e^{|t|^{\alpha}}, & t \geqslant 0 \\ e^{|t|^{\alpha}.e^{i\alpha\pi}}, & t<0. \end{cases} \\
&= \begin{cases} e^{|t|^{\alpha}}, & t \geqslant 0 \\ e^{|t|^{\alpha}(\cos\alpha\pi + i\sin\alpha\pi)}, & t<0. \end{cases} \\
&= \begin{cases} e^{|t|^{\alpha}}, & t \geqslant 0 \\ e^{|t|^{\alpha}\cos\alpha\pi}\left(\cos\left(|t|^{\alpha}\sin\alpha\pi\right) + i\sin\left(|t|^{\alpha}\sin\alpha\pi\right)\right), & t<0. \end{cases}
\end{aligned}$$

For example

$$e^{t^{\frac{1}{2}}} = \begin{cases} e^{|t|^{\frac{1}{2}}}, & t \geqslant 0 \\ \cos|t|^{\frac{1}{2}} + i\ \sin|t|^{\frac{1}{2}}, & t<0. \end{cases} = \begin{cases} e^{|t|^{\frac{1}{2}}}, & t \geqslant 0 \\ e^{i|t|^{\frac{1}{2}}}, & t<0. \end{cases}$$

Definition 2.2. (α- Differentiable function) Let $f(t)$ be a real valued function defined in an open interval containing a real number a and let a^{α}, $\alpha \in (0,1]$, denotes to the principal exponential of the real number a. The function $f(t)$ is α- differentiable at a if

$$\frac{df}{dt^{\alpha}}(a) := \lim_{t\to a} \frac{f(t)-f(a)}{t^{\alpha}-a^{\alpha}}$$

exists as a complex number. More generally, a function f is said to be $\alpha-$ differentiable on an open set Ω if it is $\alpha-$ differentiable at every point in Ω, and a $\alpha-$ differentiable function is one in which $\frac{df}{dt^{\alpha}}$ exists on its domain.

Example 2.3. If $f(t)=t$. Find $\frac{df}{dt^{\frac{1}{2}}}(t)$ at $t=1$, $t=-1$ and $t=0$.

$$\begin{aligned}
\frac{df}{dt^{\frac{1}{2}}}(a) &= \lim_{\tau\to a} \frac{\tau-a}{\tau^{\frac{1}{2}}-a^{\frac{1}{2}}} \\
&= \lim_{\tau\to a} \frac{\tau-a}{|\tau|^{\frac{1}{2}}-|a|^{\frac{1}{2}}}.\begin{cases} 1 & a \geqslant 0 \\ e^{\frac{-i\pi}{2}} & a<0 \end{cases} \\
&= \lim_{\tau\to a} \frac{\tau-a.\left(|\tau|^{\frac{1}{2}}+|a|^{\frac{1}{2}}\right)}{|\tau|-|a|}.\begin{cases} 1 & a \geqslant 0 \\ -i & a<0 \end{cases} \\
&= \begin{cases} 2|a|^{\frac{1}{2}} & a \geqslant 0 \\ 2i|a|^{\frac{1}{2}} & a<0 \end{cases}
\end{aligned}$$

So,

$$\left.\frac{df}{dt^{\frac{1}{2}}}\right|_{t=1}=2, \quad \left.\frac{df}{dt^{\frac{1}{2}}}\right|_{t=0}=0 \ \text{ and } \ \left.\frac{df}{dt^{\frac{1}{2}}}\right|_{t=-1}=2i$$

Example 2.4. Prove that $\frac{d\,f}{dt^{\frac{1}{2}}}(e^{t^{\frac{1}{2}}}) = e^{t^{\frac{1}{2}}}$

$$\begin{aligned}
\frac{d\,f}{dt^{\frac{1}{2}}}(t) &= \lim_{\tau\to t}\frac{e^{\tau^{\frac{1}{2}}} - e^{t^{\frac{1}{2}}}}{\tau^{\frac{1}{2}} - t^{\frac{1}{2}}} \\
&= \lim_{\tau\to t}\frac{e^{\tau^{\frac{1}{2}}} - e^{t^{\frac{1}{2}}}}{|\tau|^{\frac{1}{2}} - |t|^{\frac{1}{2}}}.\begin{cases} 1 & t \geqslant 0 \\ -i & t < 0 \end{cases} \\
&= \begin{cases} e^{|t|^{\frac{1}{2}}} & t \geqslant 0 \\ i\,e^{i|t|^{\frac{1}{2}}} & t < 0 \end{cases}.\begin{cases} 1 & t \geqslant 0 \\ -i & t < 0 \end{cases} \\
&= \begin{cases} e^{|t|^{\frac{1}{2}}} & t \geqslant 0 \\ e^{i|t|^{\frac{1}{2}}} & t < 0 \end{cases} = e^{t^{\frac{1}{2}}}
\end{aligned}$$

In general, we can prove that $\frac{d\,f}{dt^{\alpha}}(e^{t^{\alpha}}) = e^{t^{\alpha}}$

Remark 2.5. If $\frac{d\,f}{dt^{\alpha}}$ exists on its domain, then it is a complex-valued function given by

$$\frac{d\,f}{dt^{\alpha}} = f_{\alpha}(t).s_{\alpha}(t)$$

where $f_{\alpha}(t)$ is a real valued function defined by

$$f_{\alpha}(t) := \frac{d\,f}{d|t|^{\alpha}} = \lim_{\tau\to t}\frac{f(\tau) - f(t)}{|\tau|^{\alpha} - |t|^{\alpha}}$$

and s_{α} is a step complex valued function defined by

$$s_{\alpha}(t) = \begin{cases} 1 & t \geqslant 0 \\ e^{-i\alpha\pi} & t < 0 \end{cases}$$

Remark 2.6. For $0 < \alpha < 1$, If $\frac{d\,f}{dt^{\alpha}}$ is exist at zero, then

$$\frac{d\,f}{dt^{\alpha}}(0) = real\,constant$$

Remark 2.7. If $0 < \alpha_2 < \alpha_1 \leqslant 1$ and $f(t)$ is α_1- differentiable at a, then it is α_2- differentiable at a, moreover

$$\frac{d\,f}{dt^{\alpha_2}}(a) = \frac{\alpha_1\,a^{\alpha_1-\alpha_2}}{\alpha_2}\frac{d\,f}{dt^{\alpha_1}}(a)$$

So, if $f(t)$ is differentiable($1-$ differentiable) at a, then it is $\alpha-$ differentiable at a, and

$$\frac{d\,f}{dt^{\alpha}}(a) = \frac{a^{1-\alpha}}{\alpha}\frac{d\,f}{dt}(a)$$

The converse is not true. For example $f(t) = (t)^{\frac{1}{2}}$ is $\frac{1}{2}-$ differentiable at 0 but not differentiable at 0.

From the definition 2.2, we can easy prove the following theorems:

Theorem 2.8. *Let* $f(t)$ *and* $g(t)$ *be* $\alpha-$ *differentiable functions and* k *be a constant. Then each of the following equations holds.*

(1) $\frac{d}{dt^{\alpha}}[k] = 0$

(2) $\frac{d}{dt^{\alpha}}[k\, f(t)] = k\frac{d}{dt^{\alpha}}[f(t)]$

(3) $\frac{d}{dt^{\alpha}}[f(t) \pm g(t)] = \frac{d}{dt^{\alpha}}[f(t)] \pm \frac{d}{dt^{\alpha}}[g(t)]$

(4) $\frac{d}{dt^{\alpha}}[f(t).g(t)] = \frac{df(t)}{dt^{\alpha}}.g(t) + f(t).\frac{dg(t)}{dt^{\alpha}}$

(5) $\frac{d}{dt^{\alpha}}\left[\frac{f(t)}{g(t)}\right] = \frac{1}{(g(t))^2}\left[\frac{df(t)}{dt^{\alpha}}.g(t) - f(t).\frac{dg(t)}{dt^{\alpha}}\right], \quad g(t) \neq 0$

Theorem 2.9. *If* $y = f(x)$ *is differentiable at* x *and* $x = g(t)$ *is* $\alpha-$ *differentiable at* t*, then*

$$\frac{dy}{dt^{\alpha}} = \frac{dy}{dx}.\frac{dx}{dt^{\alpha}}$$

Theorem 2.10. *Let* $f(t)$ *be a function and* a *be in its domain. If* $f(t)$ *is* $\alpha-$ *differentiable at* a*, then it is continuous at* a*.*

3. $\alpha-$ derivative and $\alpha-$ integral operators

$\alpha-$ Derivative operators

We define the $\alpha-$ first extension derivatives operators of continuous $\alpha-$ differentiable real valued function $x = f(t)$ by

$$\frac{d^{\alpha_1}f}{dt^{\alpha_2}} := \Gamma(\alpha_1+1)\,\frac{df}{dt^{\alpha_2}} \tag{3.1}$$

$$x^{(\alpha)} = f^{(\alpha)} = D^{\alpha}f := \frac{d^{\alpha}f}{dt^{\alpha}} = \Gamma(\alpha+1)\,\frac{df}{dt^{\alpha}} \tag{3.2}$$

and if $f(t)$ is differentiable

$$\frac{d^{\alpha_1}f}{dt^{\alpha_2}} = \frac{\Gamma(\alpha_1+1)}{\alpha_2}\,t^{1-\alpha_2}\,\frac{df}{dt} \tag{3.3}$$

$$x^{(\alpha)} = f^{(\alpha)} = D^{\alpha}f = \frac{d^{\alpha}f}{dt^{\alpha}} = \Gamma(\alpha)\,t^{1-\alpha}\,\frac{df}{dt} \tag{3.4}$$

For examples $D^{\alpha}(constant) = 0, \quad D^{\alpha}\left(e^{t^{\alpha}}\right) = \Gamma(\alpha)\,e^{t^{\alpha}}, \ D^{\alpha}\left(\sin t^{\alpha}\right) = \Gamma(\alpha)\,\cos t^{\alpha}.$

We can extend the above notations to higher extension , for examples

$$x^{(\alpha_1,\alpha_2,\ldots,\alpha_n)} = D^{\alpha_1,\alpha_2,\ldots,\alpha_n}f := D^{\alpha_1}\left(D^{\alpha_2,\ldots,\alpha_n}f\right) = D^{\alpha_1,\alpha_2}\left(D^{\alpha_3},\ldots,D^{\alpha_n}f\right) \tag{3.5}$$

$$x^{(\alpha)_n} = f^{(\alpha)_n} := D^{\overbrace{\alpha,\alpha,\ldots,\alpha}^{}_{n\,times}}f \tag{3.6}$$

$$x^{(n)} = f^{(n)} = D^{n}f := x^{(1)_n} \tag{3.7}$$

$$x^{(\beta)} = f^{(\beta)} = D^{\beta}f := D^{\alpha}\left(D^{n}f\right), \quad n < \beta = \alpha+n \leqslant n+1 \tag{3.8}$$

$\alpha-$ Integral operators

If $F(t)$ is $\alpha-$ differentiable and $\dfrac{dF}{dt^{\alpha}} = f(t)$, then

$$\begin{aligned} \frac{dF}{dt^{\alpha}} = f(t) &\Leftrightarrow dF = f(t)\, dt^{\alpha} \\ &\Leftrightarrow F(t) = \int f(t)\, dt^{\alpha} \end{aligned}$$

and in general

$$\int f(t)\, dt^{\alpha} = F(t) + c$$

The integration by parts can be given by the following theorem

Theorem 3.1. *If f and g are $\alpha-$ differentiable functions, then*

$$\int \frac{df}{dt^{\alpha}}.g(t)\, dt^{\alpha} = f(t).g(t) - \int f(t).\frac{dg}{dt^{\alpha}}\, dt^{\alpha}$$

Example 3.2.

$$\begin{aligned} \int t^{\alpha}.e^{2\,t^{\alpha}}\, dt^{\alpha} &= \frac{1}{2}e^{2\,t^{\alpha}} - \int e^{2\,t^{\alpha}}\, dt^{\alpha} \\ &= \frac{1}{2}\, t^{\alpha}\, e^{2\,t^{\alpha}} - \frac{1}{4}e^{2\,t^{\alpha}} + c \end{aligned}$$

For more, see table 1 and table 2, the comparison between the definition of α calculus with the concept of Riemann-Liouville and our concept.

4. $\alpha-$ differential equation and canonical form

Definition of $\alpha-$ differential equation

Definition 4.1. ($\alpha-$ differential equation) An equation containing the $\alpha-$derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be an $\alpha-$ differential equation.

Definition 4.2. The extension of an $\alpha-$ differential equation is the largest number of recurrence derivatives in the equation. For example,

$$\underbrace{z^{(1,\frac{1}{2},1)}(t)}_{\text{3 recurrence derivatives}} - \overbrace{(5+4i)z^{(\frac{1}{3})_2}(t)}^{\text{2 recurrence derivatives}} + 7z^{(\frac{2}{3})} = 0$$

is a third extension $\alpha-$ differential equation

Definition 4.3. Any complex parameterized curve γ described by

$$z(t) = x(t) + y(t)\, i, \quad t \in \Omega \subseteq \Re$$

which when substituted into an α differential equation reduces the equation to an identity, is said to be a solution of the equation on Ω.

Table 1: Comparison between the definition of α derivative with the concept of Riemann-Liouville and our concept

Comparisons	Our Concept	Riemann-Liouville Concept
Derivative operator D^{α}, $0 < \alpha \leqslant 1$	One definition for all $t \in \Re$ $(D^{\alpha} f(t)) := \Gamma(\alpha+1).\lim\limits_{\tau \to t} \frac{f(\tau)-f(t)}{\tau^{\alpha}-t^{\alpha}}$. If f is differentiable, then $(D^{\alpha} f(t)) := \Gamma(\alpha).t^{1-\alpha}f'(t)$	Two definitions (left and right) for all $t \in [t_0, t_f]$ $(\underline{D}^{\alpha} f(t)) := \frac{1}{\Gamma(\alpha)}.\frac{d}{dt}\left(\int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau)d\tau\right)$ (left), $\left(\overline{D}^{\alpha} f(t)\right) := \frac{1}{\Gamma(\alpha)}.\frac{d}{dt}\left(\int_{t}^{t_f} (\tau-t)^{\alpha-1} f(\tau)d\tau\right)$ (right)
$\mathbf{D}^{\alpha}$ (constant)	$D^{\alpha}(constant) = 0$	$D^{\alpha}(constant) \neq 0$
Product Rule	$D^{\alpha}(u.v) = D^{\alpha}(u).v + u.D^{\alpha}(v)$	$D^{\alpha}(u.v) \neq D^{\alpha}(u).v + u.D^{\alpha}(v)$
$\mathbf{D}^{\alpha}$ **function**	It is a complex valued function	He did not explain the result of $D^{\alpha} f(t)$ when t is negative or what mean by $(t-\tau)^{\alpha-1}$ when $t-\tau < 0$
When $\alpha = 1$	It is completely identical to classical calculus when $\alpha = 1$.	It does not coincide with classical calculus when $\alpha = 1$.
Higher derivative operator $\mathbf{D}^{\beta}$, $\beta = \alpha + n$	One definition for all $t \in \Re$ $(D^{\beta} f(t)) := D^{\alpha}\left(\frac{d^n f(t)}{dt^n}\right)$	Two definitions (left and right) for all $t \in [t_0, t_f]$ $(\underline{D}^{\beta} f(t)) := \frac{1}{\Gamma(\alpha)}.\frac{d^{n+1}}{dt^{n+1}}\left(\int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau)d\tau\right)$, $\left(\overline{D}^{\alpha} f(t)\right) := \frac{1}{\Gamma(\alpha)}.\frac{d^{n+1}}{dt^{n+1}}\left(\int_{t}^{t_f} (\tau-t)^{\alpha-1} f(\tau)d\tau\right)$

Table 2: Comparison between the definition of α integral with the concept of Riemann-Liouville and our concept

Comparisons	**Our Concept**	**Riemann-Liouville Concept**
Integral operator I^{α}, $0 < \alpha \leqslant 1$	For all $t \in [t_0, t_f]$ $(I^{\alpha} f(t)) := \frac{1}{\Gamma(\alpha+1)} \cdot \lim_{n\to\infty} \lim_{\lvert P\rvert \to 0} \sum_{i=1}^{n} f(t_i^*) \left(t_{i-1}^{\alpha} - t_i^{\alpha}\right)$, Where $P = \{t_0, t_1, \ldots, t_n = t_f\}$ is any partition of $[t_0, t_f]$. For a continuous function f on $[t_0, t_f]$, we define lower and upper integral operator for $t \in [t_0, t_f]$. by: $(\underline{I}^{\alpha} f(t)) := \frac{1}{\Gamma(\alpha)} \left(\int_{t_0}^{t} (\tau)^{\alpha-1} f(\tau) d\tau \right)$, $\left(\bar{I}^{\alpha} f(t)\right) := \frac{1}{\Gamma(\alpha)} \left(\int_{t}^{t_f} (\tau)^{\alpha-1} f(\tau) d\tau \right)$	Two definitions (left and right) for all $t \in [t_0, t_f]$ $(\underline{I}^{\alpha} f(t)) := \frac{1}{\Gamma(\alpha)} \left(\int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau \right)$, $\left(\bar{I}^{\alpha} f(t)\right) := \frac{1}{\Gamma(\alpha)} \left(\int_{t}^{t_f} (\tau - t)^{\alpha-1} f(\tau) d\tau \right)$
I^{α} **function**	It is a complex valued function	He did not explain the result of $I^{\alpha} f(t)$ when t is negative or what mean by $(t-\tau)^{\alpha-1}$ when $t - \tau < 0$
Higher integral operator I^{β}, $\beta = \alpha + n$	For a continuous function f on $[t_0, t_f]$, we define lower and upper integral operator for $t \in [t_0, t_f]$, by: $\left(\underline{I}^{\beta} f(t)\right) := \frac{1}{\Gamma(\alpha)} \left(\int_{t_0}^{t} (t-\tau)^{n-1} (\tau)^{\alpha} f(\tau)\, d\tau \right)$, $\left(\bar{I}^{\beta} f(t)\right) := \frac{1}{\Gamma(\alpha)} \left(\int_{t}^{t_f} (\tau - t)^{n-1} (\tau)^{\alpha} f(\tau)\, d\tau \right)$	Two definitions (left and right) for all $t \in [t_0, t_f]$ $\left(\underline{I}^{\beta} f(t)\right) := \frac{1}{\Gamma(\alpha)} \left(\int_{t_0}^{t} (t-\tau)^{n+\alpha-1} f(\tau) d\tau \right)$, $\left(\bar{I}^{\beta} f(t)\right) := \frac{1}{\Gamma(\alpha)} \left(\int_{t}^{t_f} (\tau - t)^{n+\alpha-1} f(\tau) d\tau \right)$

Example 4.4. Verify that

$$z(t) = t^2 + 2t\,i$$

is a solution of the fractional differential equation

$$2z^{(\frac{1}{2})_2}(t) - 3\,\pi z^{(1)}(t) = 0 \quad \text{on } \mathfrak{R}.$$

$$z^{(\frac{1}{2})}(t) = \Gamma(\frac{3}{2})\,\frac{2}{\frac{1}{2}}\,t^{\frac{3}{2}} + 2\,\Gamma(\frac{3}{2})\,\frac{1}{\frac{1}{2}}\,t^{\frac{1}{2}}\,i = 2\sqrt{\pi}\,t^{\frac{3}{2}} + 2\sqrt{\pi}\,t^{\frac{1}{2}}\,i$$

$$z^{(\frac{1}{2})_2}(t) = D^{\frac{1}{2}}\left(z^{(\frac{1}{2})}(t)\right) = 3\,\pi\,t + \pi\,i$$

then

$$2z^{(\frac{1}{2})_2}(t) - 3\,\pi z^{(1)}(t) = 6\,\pi\,t + 2\pi\,i\; - 6\,\pi\,t - 2\pi\,i\; = 0$$

Canonical form

From remark 2.7, if $\alpha, \alpha_1 \in (0,1]$, $\alpha > \alpha_1$, and $z(t)$ is $\alpha-$ differentiable, then $z^{(\alpha_1)} = \dfrac{\Gamma(\alpha_1)}{\Gamma(\alpha)}\,t^{\alpha-\alpha_1}\,z^{(\alpha)}$. So, we can transfer any α differential equation containing $z^{(\alpha_1)}$, $z^{(\alpha)}$ to another equivalent equation containing $z^{(\alpha)}$ only.

For example, if a_0, a_i, $a, u : [t_0, t_f] \to \mathfrak{C}$ are given continuous functions, α, $\alpha_i \in (0,1]$ and $\alpha > \alpha_i$ for all $i = 1, 2, .., m$, then the linear α differential equation of first extension

$$a_0(t)\,z^{(\alpha)}(t) + a_1(t)\,z^{(\alpha_1)}(t) + \cdots + a_m(t)\,z^{(\alpha_m)}(t) = a(t)\,z(t) + u(t), \tag{4.1}$$

can be transform to the canonical form

$$p(t)\,z^{(\alpha)}(t) = a(t)\,z(t) + u(t), \tag{4.2}$$

where

$$p(t) = \left[a_0(t) + \frac{\Gamma(\alpha_1)}{\Gamma(\alpha)}\,a_1(t)\,t^{\alpha-\alpha_1} + \cdots + \frac{\Gamma(\alpha_m)}{\Gamma(\alpha)}\,a_m(t)\,t^{\alpha-\alpha_m}\right]$$

In general the canonical form of the linear α differential equation of extension n can be taken the form

$$p_n(t)\,z^{(\alpha)_n}(t) + p_{n-1}(t)\,z^{(\alpha)_{n-1}}(t) + \cdots + p_1(t) z^{(\alpha)}(t) = a(t)\,z(t) + u(t) \tag{4.3}$$

Example 4.5. Transfer the following second extension equation to the canonical form;

$$z^{(1,\frac{1}{2})}(t) + 4\,z^{(\frac{1}{2})}(t) = 2\,z(t) + 5 \tag{4.4}$$

Here $\alpha = 1$ is the largest one, so

$$z^{(\frac{1}{2})}(t) = \frac{\Gamma(\frac{1}{2})}{\Gamma(1)}\,t^{1-\frac{1}{2}}\,z^{(1)}(t) = \sqrt{\pi}t^{\frac{1}{2}}\,z^{(1)}(t)$$

$$z^{(1,\frac{1}{2})}(t) = D\left(z^{(\frac{1}{2})}(t)\right) = D\left(\sqrt{\pi}t^{\frac{1}{2}}\,z^{(1)}(t)\right) = \sqrt{\pi}t^{\frac{1}{2}}\,z^{(1)_2}(t) + \frac{1}{2}\sqrt{\pi}t^{-\frac{1}{2}}\,z^{(1)}(t)$$

then the canonical form of (4.4) is

$$\sqrt{\pi}\left[t^{\frac{1}{2}}\, z^{(1)_2}(t) + \left(4\, t^{\frac{1}{2}} + \frac{1}{2} t^{-\frac{1}{2}}\right) z^{(1)}(t)\right] = 2\, z(t) + 5$$

i.e.

$$p_1(t) = \sqrt{\pi}\left(4\, t^{\frac{1}{2}} + \frac{1}{2} t^{-\frac{1}{2}}\right), \qquad p_2(t) = \sqrt{\pi}\, t^{\frac{1}{2}}$$

5. Linear α− equations of first extension

We start with a solution of a homogeneous linear α− equations of first extension. Then we introduce a nonhomogeneous and initial value problems cases. Let a be a given complex number. Then,

$$D^{\alpha}\left(e^{\frac{a}{\Gamma(\alpha+1)}\, t^{\alpha}}\right) = \Gamma(\alpha+1)\frac{d}{dt^{\alpha}}\left(e^{\frac{a}{\Gamma(\alpha+1)}\, t^{\alpha}}\right) = a\, e^{\frac{a}{\Gamma(\alpha+1)}\, t^{\alpha}}$$

This shows that

$$z(t) = C\, e^{\frac{a}{\Gamma(\alpha+1)}\, t^{\alpha}} \tag{5.1}$$

is the general solution of the homogeneous equation

$$z^{(\alpha)}(t) = az(t) \tag{5.2}$$

and if a is a continuous function $a : [t_0, t_f] \to \mathcal{C}$, then

$$z(t) = C\, e^{\frac{1}{\Gamma(\alpha+1)}} \int a(t)dt^{\alpha} \tag{5.3}$$

is the general solution of the homogeneous equation

$$z^{(\alpha)}(t) = a(t)z(t) \tag{5.4}$$

If $z(t_0) = z_0$, then the solution is unique and given by

$$z(t) = z_0\, e^{\frac{1}{\Gamma(\alpha+1)}} \int a(t)dt^{\alpha} \tag{5.5}$$

Example 5.1. Solve

$$z^{(\alpha)}(t) = z(t), \quad z(0) = 1, \quad t \in \mathfrak{R}$$

the solution is given by

$$\begin{aligned} z(t) &= e^{\frac{1}{\Gamma(\alpha+1)}\, t^{\alpha}} \\ &= e^{\frac{1}{\Gamma(\alpha+1)}\, t^{\alpha}} \\ &= \begin{cases} e^{\frac{1}{\Gamma(\alpha+1)}|t|^{\alpha}}, & t \geqslant 0 \\ e^{\frac{1}{\Gamma(\alpha+1)}\, i\,|t|^{\alpha}}, & t < 0. \end{cases} \\ &= x(t) + y(t)i \end{aligned}$$

where (see Fig. 1)

$$x(t) = \begin{cases} e^{\frac{1}{\Gamma(\alpha+1)}|t|^{\alpha}}, & t \geqslant 0 \\ \cos \frac{1}{\Gamma(\alpha+1)}|t|^{\alpha}, & t < 0. \end{cases}, \quad y(t) = \begin{cases} 0, & t \geqslant 0 \\ \sin \frac{1}{\Gamma(\alpha+1)}|t|^{\alpha}, & t < 0. \end{cases}$$

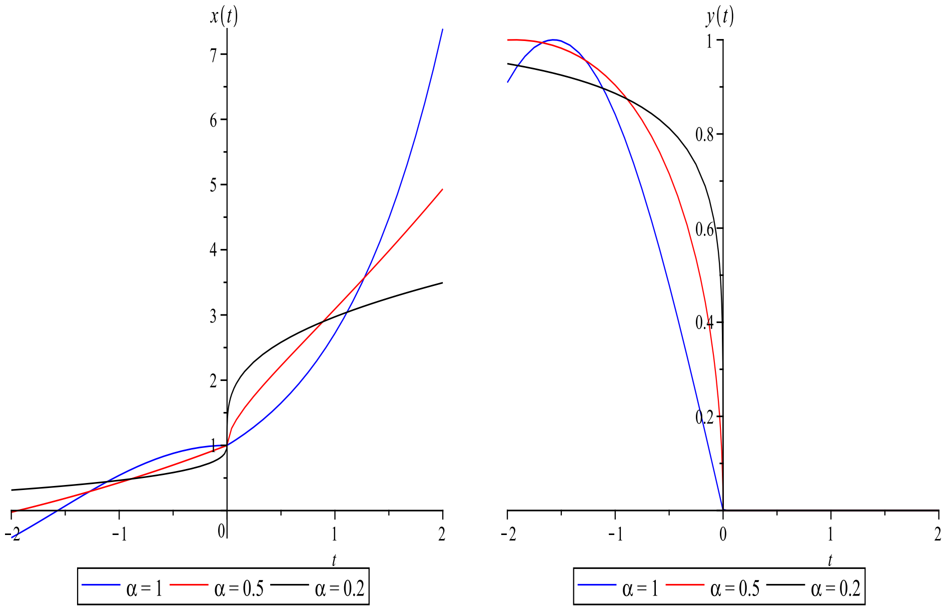

Figure 1: solution of example 5.1

We now construct the formula for the solutions of a nonhomogeneous linear α differential equations in a canonical form

Theorem 5.2. *Let* $p,\ a,\ u:\Omega\to\mathbb{C}$ *be continuous functions on* Ω*, and let* $p(t)\neq 0$ *for all* $t\in\ \Omega$*. Then, the* α *differential equation of first extension*

$$p(t)\,z^{(\alpha)}(t)=a(t)\,z(t)+u(t) \tag{5.6}$$

has infinitely many solutions given by

$$z(t)=C\,\Psi(t)+\frac{1}{\Gamma(\alpha+1)}\Psi(t)\int\Psi(-t)\,u(t)\,dt^{\alpha}$$

where

$$\Psi(t)=e^{\frac{1}{\Gamma(\alpha+1)}\int p^{-1}(t)\,a(t)dt^{\alpha}}$$

Proof. Write the differential equation with z on one side only,

$$z^{(\alpha)}(t)-\bar{a}(t)\,z(t)=u(t),\quad \bar{a}(t)=p^{-1}(t)\,a(t) \tag{5.7}$$

and then multiply the differential equation by a function $\mu(t)$, called an $\alpha-$integrating factor,

$$\mu(t)z^{(\alpha)}(t)-\mu(t)\,\bar{a}(t)\,z(t)=\mu(t)\,u(t) \tag{5.8}$$

We choose a function $\mu(t) = e^{\frac{-1}{\Gamma(\alpha+1)}\int \bar{a}(t)dt^{\alpha}}$ solution of

$$-\bar{a}(t)\,\mu(t) = \mu^{(\alpha)}(t) \tag{5.9}$$

For any function $\mu(t)$ solution of Eq. (5.9), the differential equation in (5.8) has the form

$$\mu(t)z^{(\alpha)}(t) - \mu^{(\alpha)}(t)\,z(t) = \mu(t)\,u(t) \tag{5.10}$$

But the left-hand side is a total $\alpha-$ derivative of a product of two functions,

$$(\mu(t)z(t))^{(\alpha)} = \mu(t)\,u(t)$$

then

$$\mu(t)z(t) = C + \frac{1}{\Gamma(\alpha+1)}\int \mu(t)\,u(t)\,dt^{\alpha}$$

or

$$z(t) = C\,\mu^{-1}(t) + \frac{1}{\Gamma(\alpha+1)}\,\mu^{-1}(t)\int \mu(t)\,u(t)\,dt^{\alpha}$$

$\square$

Remark 5.3. Although the $alpha$ differential equation in (5.7) has infinitely many solutions, the associated initial value problem has a unique solution.If $z(t_0) = z_0$, z_0 is a given point in $\mathcal{C}$, then the solution is unique and is given by the formula

$$z(t) = \Psi(t)\,z_0 + \frac{1}{\Gamma(\alpha+1)}\int_{t_0}^{t}\Psi(t-\tau)\,u(\tau)\,d\tau^{\alpha}$$

where

$$\Psi(t) = e^{\frac{1}{\Gamma(\alpha+1)}\int_{t_0}^{t} p^{-1}(t)\,a(t)dt^{\alpha}}$$

6. Linear $\alpha-$ first extension system

If $z_1(t), z_2(t), \ldots z_n(t)$ are $\alpha-$ differentiable, then any $n-$ nonhomogenous system of first extension can be transfer to canonical form:

$$P(t)\begin{bmatrix} z_1(t)\\ z_2(t)\\ \vdots\\ z_n(t)\end{bmatrix}^{(\alpha)} = A(t)\begin{bmatrix} z_1(t)\\ z_2(t)\\ \vdots\\ z_n(t)\end{bmatrix} + \begin{bmatrix} u_1(t)\\ u_2(t)\\ \vdots\\ u_n(t)\end{bmatrix} \tag{6.1}$$

where, $A(t), P(t)$ are $n\times n$ matrices and $[u_1, u_2, \ldots, u_n]^T$ is $n-$ vector. Consider the initial condition is given by

$$\begin{bmatrix} z_1(t_0)\\ z_2(t_0)\\ \vdots\\ z_n(t_0)\end{bmatrix} = \begin{bmatrix} z_{10}\\ z_{20}\\ \vdots\\ z_{n0}\end{bmatrix} \tag{6.2}$$

Theorem 6.1. *Let* $A(t)$, $P(t)$, $u(t)$ *be complex continuous functions defined on* $\Omega \subseteq \Re$ *and let* $P(t)$ *invertible for all* $t \in \Omega$. *Then the initial value problem* (6.1)-(6.2) *has the unique solution* z *on the domain* Ω, *given by*

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \Psi(t) \begin{bmatrix} z_{10} \\ z_{20} \\ \vdots \\ z_{n0} \end{bmatrix} + \frac{1}{\Gamma(\alpha+1)} \int_{t_0}^{t} \Psi(t-\tau) \begin{bmatrix} u_1(\tau) \\ u_2(\tau) \\ \vdots \\ u_n(\tau) \end{bmatrix} d\tau^{\alpha}$$

where

$$\Psi(t) = e^{\frac{1}{\Gamma(\alpha+1)} \int_{t_0}^{t} P^{-1}(\tau) A(\tau) d\tau^{\alpha}},$$

Example 6.2. Consider the following system

$$\begin{cases} z_1^{(1)}(t) = z_2(t) \\ z_2^{(1)}(t) + z_1^{\alpha}(t) = z_1(t), \\ z_1(0) = 1, \ z_2(0) = 1 \end{cases} \tag{6.3}$$

we can transfer (6.3)to canonical form:

$$\begin{cases} z_1^{(1)}(t) = z_2(t) \\ z_2^{(1)}(t) + \Gamma(\alpha)\, t^{1-\alpha}\, z_1^{(1)}(t) = z_1(t) \\ z_1(0) = 1, \ z_2(0) = 1 \end{cases} \tag{6.4}$$

in the matrix form

$$\begin{bmatrix} 1 & 0 \\ \Gamma(\alpha)\, t^{1-\alpha} & 1 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \tag{6.5}$$

with $[z_1(0), z_2(0)]^T = [1,1]^T$, then the solution(Fig. 2) is given by

$$\begin{aligned} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} &= e^{\int \begin{bmatrix} 1 & 0 \\ \Gamma(\alpha)\, t^{1-\alpha} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} dt} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= e^{\int \begin{bmatrix} 0 & 1 \\ 1 & -\Gamma(\alpha)\, t^{1-\alpha} \end{bmatrix} dt} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= e^{\begin{bmatrix} 0 & t \\ t & -\frac{\Gamma(\alpha)}{(2-\alpha)}\, t^{2-\alpha} \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned} \tag{6.6}$$

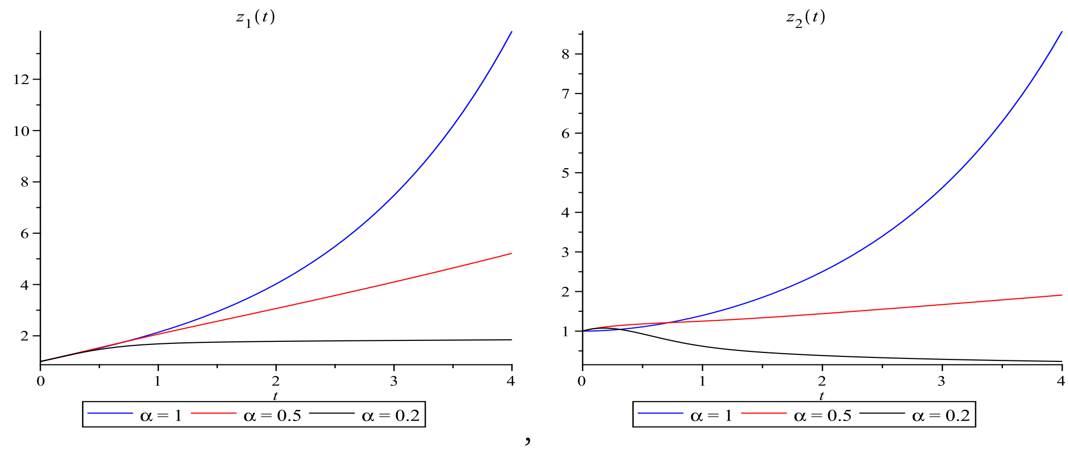

Figure 2: solution of example 6.2

7. Canonical form with constant coefficients

If $P(t) = P$, $A(t) = A$ are constant matrices and P is invertible, then $\alpha-$ differentiable can be taken the form:

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix}^{(\alpha)} = A \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix} \tag{7.1}$$

where, A is constant $n \times n$ matrix and $[u_1, u_2, \dots, u_n]^T$ is $n-$ vector. Consider the initial condition is given by

$$\begin{bmatrix} z_1(t_0) \\ z_2(t_0) \\ \vdots \\ z_n(t_0) \end{bmatrix} = \begin{bmatrix} z_{10} \\ z_{20} \\ \vdots \\ z_{n0} \end{bmatrix} \tag{7.2}$$

Theorem 7.1. *The system (7.1)-(7.2) has the unique solution z on the domain $[t_0, t_f]$, given by*

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} = e^{\frac{t^\alpha - t_0^\alpha}{\Gamma(\alpha+1)}} \begin{bmatrix} z_{10} \\ z_{20} \\ \vdots \\ z_{n0} \end{bmatrix} + \frac{1}{\Gamma(\alpha+1)} \int_{t_0}^{t} e^{\frac{t^\alpha - \tau^\alpha}{\Gamma(\alpha+1)}} \begin{bmatrix} u_1(\tau) \\ u_2(\tau) \\ \vdots \\ u_n(\tau) \end{bmatrix} d\tau^\alpha$$

Example 7.2. Solve the following differential system:

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}^{(\alpha)} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ e^{\frac{2t^\alpha}{\Gamma(\alpha+1)}} \end{bmatrix},$$

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

the solution(Fig. 3) is given by

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = e^{\frac{1}{\Gamma(\alpha+1)}\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} t^{\alpha}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{\Gamma(\alpha+1)} \int_0^t e^{\frac{1}{\Gamma(\alpha+1)}\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}(t^{\alpha}-\tau^{\alpha})} \begin{bmatrix} 0 \\ e^{\frac{2t^{\alpha}}{\Gamma(\alpha+1)}} \end{bmatrix} d\tau^{\alpha}$$

$$e^{\frac{1}{\Gamma(\alpha+1)}\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} t^{\alpha}} = \frac{1}{\Gamma(\alpha+1)} e^{\frac{2t^{\alpha}}{\Gamma(\alpha+1)}} \begin{bmatrix} \Gamma(\alpha+1)+t^{\alpha} & t^{\alpha} \\ -t^{\alpha} & \Gamma(\alpha+1)-t^{\alpha} \end{bmatrix}$$

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \frac{1}{\Gamma(\alpha+1)} e^{\frac{t^{2\alpha}}{\Gamma(\alpha+1)}} \begin{bmatrix} \Gamma(\alpha+1)+2t^{\alpha}+\frac{1}{2\Gamma(\alpha+1)}t^{2\alpha} \\ \Gamma(\alpha+1)-t^{\alpha}-\frac{1}{2\Gamma(\alpha+1)}t^{2\alpha} \end{bmatrix}$$

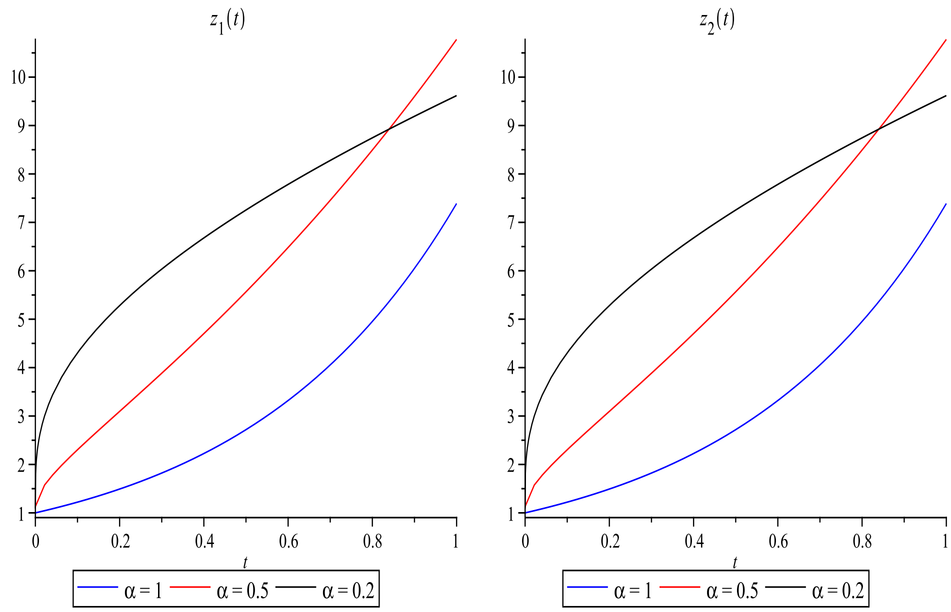

Figure 3: solution of example 7.2

Example 7.3. Consider the system

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix}^{(\alpha)} = A \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & -9 \\ 3 & 6 & 1 \end{bmatrix},$$

$$\begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}$$

The eigenvalues of the matrix A are $\lambda_1 = -1, \quad \lambda_2 = 1+3\sqrt{6}i, \quad \lambda_3 = 1-3\sqrt{6}i$ and their corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 58 \\ -31 \\ 6 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ \sqrt{6}i \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ -\sqrt{6}i \\ 2 \end{bmatrix},$$

respectively. Therefore, the solution(Fig. 4) is given by

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix} = \begin{bmatrix} e^{\frac{-t^\alpha}{\Gamma(\alpha+1)}} \\ \frac{31}{58}\, e^{\frac{t^\alpha}{\Gamma(\alpha+1)}} \cos\left(\frac{3\sqrt{6}\,t^\alpha}{\Gamma(\alpha+1)}\right) - \frac{9\sqrt{6}}{58}\, e^{\frac{t^\alpha}{\Gamma(\alpha+1)}} \sin\left(\frac{3\sqrt{6}\,t^\alpha}{\Gamma(\alpha+1)}\right) + \frac{93}{58}\, e^{\frac{-t^\alpha}{\Gamma(\alpha+1)}} \\ \frac{9}{29}\, e^{\frac{t^\alpha}{\Gamma(\alpha+1)}} \cos\left(\frac{3\sqrt{6}\,t^\alpha}{\Gamma(\alpha+1)}\right) + \frac{197\sqrt{6}}{174}\, e^{\frac{t^\alpha}{\Gamma(\alpha+1)}} \sin\left(\frac{3\sqrt{6}\,t^\alpha}{\Gamma(\alpha+1)}\right) - \frac{9}{29}\, e^{\frac{-t^\alpha}{\Gamma(\alpha+1)}} \end{bmatrix}$$

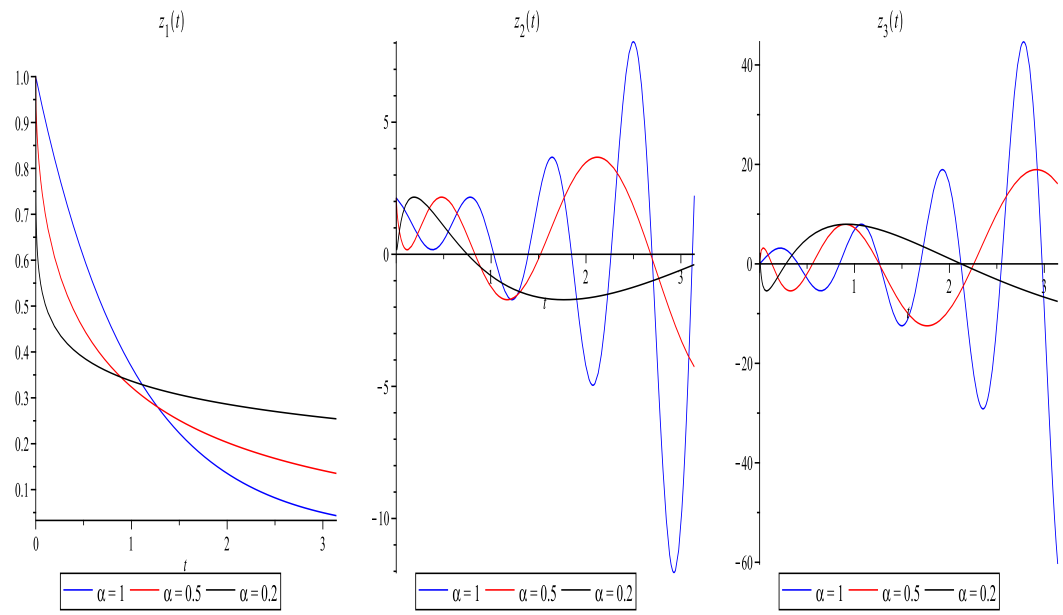

Figure 4: solution of example 7.3

8. α− Linear Higher Extension Equation

Here, we are interested in finding solutions for initial value problem of higher extension equation , we transform scalar equations of extension n into a n dimensional system of first-order equations. Consider the equation

$$z^{(\alpha)_n}(t) + p_{n-1}(t)\, z^{(\alpha)_{n-1}}(t) + \cdots + p_1(t) z^{(\alpha)}(t) = a(t)\, z(t) + u(t) \tag{8.1}$$

with initial conditions

$$z(t_0) = z_0, \quad z^{(\alpha)}(t_0) = z_1, \ldots, z^{(\alpha)_{n-1}}(t_0) = z_{n-1} \tag{8.2}$$

This equations may be written as a system of first-order equations of dimension n. We let

$$\begin{aligned} z_0(t) &= z(t) \\ z_1(t) &= z_0^{(\alpha)}(t) \\ \vdots &= \vdots \\ z_{n-1}(t) &= z_{n-2}^{(\alpha)}(t) \end{aligned}$$

Hence (8.1)-(8.2) are transfer to the system

$$\begin{aligned} z_0^{(\alpha)}(t) &= z_1(t), \qquad z_0(t_0) = z_0 \\ \vdots &= \vdots \\ z_{n-2}^{(\alpha)}(t) &= z_{n-1}(t), \qquad z_{n-2}(t_0) = z_{n-2} \\ z_{n-1}^{(\alpha)}(t) &= -p_{n-1}z_{n-1}(t) - \cdots - p_1 z_1(t) + a(t)\, z(t) + u(t), \qquad z_{n-1}(t_0) = z_{n-1} \end{aligned}$$

Let $\mathbf{z}(t) = [z_0(t), z_1(t), \ldots, z_{n-1}(t)]^T$, in vector notation, we transcribe this system as

$$\mathbf{z}^{(\alpha)}(t) = A(t)\,\mathbf{z}(t) + \mathbf{u}(t), \qquad \mathbf{z}(t_0) = \mathbf{z}_0 \tag{8.3}$$

where

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ a(t) & -p_1(t) & -p_2(t) & \ldots & -p_{n-1}(t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ u(t) \end{bmatrix}, \quad \mathbf{z}_0 = \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-2} \\ z_{n-1} \end{bmatrix}$$

If $A(t)$ is continuous for all $t \in \Omega \subseteq \Re$, then the system (8.1) -(8.2) has a unique solution given by

$$\mathbf{z}(t) = \Psi(t)\,\mathbf{z}_0 + \frac{1}{\Gamma(\alpha+1)} \int_{t_0}^{t} \Psi(t-\tau)\,\mathbf{u}(\tau)\, d\tau^{\alpha}$$

where

$$\Psi(t) = e^{\frac{1}{\Gamma(\alpha+1)}\int_{t_0}^{t} A(\tau) d\tau^{\alpha}},$$

and if $A(t) = A(\text{constant})$, then

$$\Psi(t) = e^{\frac{t^{\alpha}-t_0^{\alpha}}{\Gamma(\alpha+1)}}$$

Example 8.1. Consider the system

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix}^{(\alpha)} = A \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

the solution (Fig 5) is given by

$$z(t) = \frac{1}{3}\left[\cosh\frac{\sqrt{2}\,t^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{\sqrt{2}}\sinh\frac{\sqrt{2}\,t^{\alpha}}{\Gamma(\alpha+1)} - \cos\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \sin\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right]$$

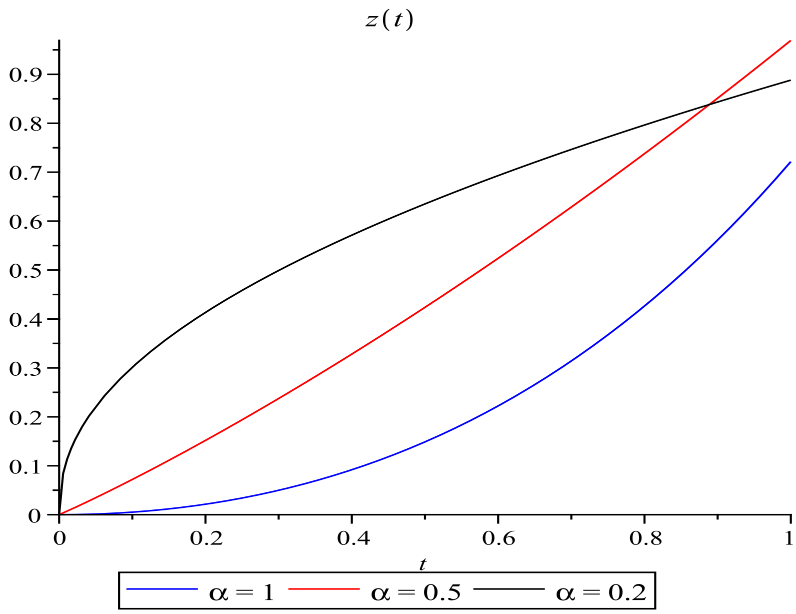

Figure 5: solution of example 8.1

9. Extended to α– partial differential equations

Definition 9.1. (α- Partial differentiation) If $f(t,x)$ is a real valued function of two variables t and x, then $\frac{\partial f}{\partial t^{\alpha}}$ and $\frac{\partial f}{\partial x^{\alpha}}$ are defined respectively by

$$\frac{\partial f}{\partial t^{\alpha}} := \lim_{\tau\to t}\frac{f(\tau,x)-f(t,x)}{\tau^{\alpha}-t^{\alpha}}$$

$$\frac{\partial f}{\partial x^{\alpha}} := \lim_{\xi\to x}\frac{f(t,\xi)-f(t,x)}{\xi^{\alpha}-x^{\alpha}}$$

Example 9.2. (α- Heat equation) Consider a curve γ in C^2 parameterized by:

$$z(t,s) = x(t,s) + iy(t,s), \quad 0 < s < L, \quad t > 0,.$$

We can construct the formal solution for initial and boundary value problem for α heat equation:

$$\frac{\partial z}{\partial t^{\alpha}} = c^2\frac{\partial^2 z}{\partial s^2},$$
$$z(t,0) = 0, \quad z(t,L) = 0, \quad t > 0,$$
$$z(0,s) = g(s), \quad 0 < s < L$$

By using separation of variable method, we get the solution

$$z(t,s)=\sum_{n=1}^{\infty} C_n\, e^{-\left(\frac{n\pi c}{L}\right)^2 t^{\alpha}} \sin\frac{n\pi s}{L}$$

where

$$C_n=\frac{2}{L}\int_0^L g(s)\sin\frac{n\pi s}{L}\,ds, \qquad n=1,2,\ldots$$

If $c=L=1$, and $g(s)=\sin\pi s$, then we have $C_1=1$ and all other $C_n=0$, so the solution is

$$z(t,s)=e^{-\pi^2 t^{\alpha}}\sin\pi s$$

10. Conclusion

In this paper, the general $\alpha-$ differential equation (based on Shehata's definition of α differentiation and α integration) was studied and the exact solution formula of first and higher extension linear α differential equations defined in complex parameterized curve was obtained. The results we have got here is analogous to the results obtained from the classical case when $\alpha=1$.

Declarations

Conflict of interest: The author declare that they have no conflict of interest.

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