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Dynamics of diverse nonlinear water waves to a fifth-order nonlinear soliton model

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Abstract

The objective of this study is to examine the dynamic behavior of the fifth-order nonlinear equation (FONLE) and its practical application in the field of applied mathematics. To achieve this, we first employ a traveling wave transformation technique to convert the FONLE equation into a nonlinear ordinary differential equation. Various exact soliton solutions of FONLE are then obtained using the $\left(\frac{G'}{G^2}\right)$-expansion method and the $\exp(-\phi(\kappa))$ method. These soliton solutions can be utilized in numerous areas of mathematical physics and nonlinear sciences, including ocean engineering, optical fibers, plasma physics, applied mathematics, and fluid dynamics. The techniques used in this study are applied to this model for the first time.

Keywords: $\left(\frac{G'}{G^2}\right)$-expansion method; $\exp(-\phi(\kappa))$ method ; Nonlinear Differential equation; Solitary wave solutions.

1. Introduction

Nonlinear phenomena occur in all areas of science, such as mathematics, physics, chemistry, and engineering [1]. In most applications, nonlinear systems are modelled by partial differential equations called nonlinear evolution equations (NLEEs) of much interest are solutions in the shape of spatially-localized excitations. Many such excitations are non-dispersive and non-dissipative waves. The configurations that are weak or non-radiative and that preserve their shape as they propagate to significantly long distances. Due to this special characteristic, these configurations are known as solitons or solitary waves. Soliton-like solutions occur in diverse research areas, including metamaterials, hydrodynamics, nuclear physics plasma physics, nonlinear optics, optical communications, and astrophysics [2, 3]. Many mathematicians utilized different techniques to find the exact traveling wave solution such as the Sine-Gordon expansion approach[5, 6], the Jacobi-elliptic technique [4], the homogeneous balance technique[9],

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the modified simple equation technique[7, 8], the Kudryashov technique [13], the auxiliary equation technique[10, 11], the exp-function technique[14, 25], the extended direct algebraic technique[12], the extended tanh expansion technique [24], $\left(\frac{G'}{G}\right)$-expansion technique[19, 20],$\left(\frac{G'}{G^2}\right)$-expansion technique [31], $\exp(-\phi(\tau))$ technique [16], Auto-Bäcklund transformation [17], the Hirota's bilinear approach [18], and many more.
The prime task of this research is to consider the following nonlinear water wave equation,

$$v_t(x,t)+v_x(x,t)+c_1v_x(x,t)+c_2v_{xxx}(x,t)+c_3v_x(x,t)v_{xx}(x,t)+c_4v(x,t)v_{xxx}(x,t) + c_5v_{xxxx}(x,t)=0. \tag{1.1}$$

and examine the dynamics of its soliton through applying the $\left(\frac{G'}{G^2}\right)$-expansion method and $\exp(-\phi(\kappa))$ method. In Eq. (1.1) c_1, c_2, c_3, c_4, c_5 are the dispersion term and c_3, c_4 are the non-linear terms. Many techniques have been applied to the NLWWE: W-shaped and other soliton solutions of FONLE were obtained by using the Kudryashov technique [21, 22]. The approximate solutions of NLWWE were attained by utilizing the homotopy analysis technique [23]. The stability analysis for the NLWWE was investigated by using a linear stability scheme [25]. The solitary wave solutions of NLWWE were investigated by using modified extended tanh-technique [26]. The exact solution of coupled KdV–Schrodinger equations has been attained by utilizing the Kudryashov-expansion method [27]. By using the unified method the traveling wave solution of the nonlinear fractional equation was attained in [28]. Hyperbolic and periodic waves of nonlinear NLWWE were attained by utilizing bilinear Bäcklund transformation [29]. The rogue wave solutions of the Boussinesq equation were achieved by employing Bell polynomial and Hirota's bilinearization method [30]. In the past $\left(\frac{G'}{G^2}\right)$-expansion and $\exp(-\phi(\kappa))$ method have been applied on different models: The traveling wave solutions of Time-fractional Burgers equation were attained by using $\left(\frac{G'}{G^2}\right)$-expansion method [31]. The rational, periodic, and hyperbolic functions of Classical Boussinesq equation were obtained by utilizing $\left(\frac{G'}{G^2}\right)$-expansion method [32]. The variety of exact solutions of fractional mKdV equations were obtained by utilizing $\exp(-\phi(\kappa))$ method [16]. The solitary wave solutions 0f Lakshmanan–Porsezian–Daniel (LPD) model were obtained by utilizing $\exp(-\phi(\kappa))$ method [33]. The different structures of soliton solutions of perturbed Chen–Lee–Liu equation were attained by utilizing $\exp(-\phi(\kappa))$ method[34]. The exact solutions of Gerdjikov–Ivanov equation were attained by utilizing $\exp(-\phi(\kappa))$ method [35].
In this paper we have used nonlinear differential equation with two different method to obtained the variety of soliton solutions. In section(2) we are present the description of methods. In section(3) the analysis of equation. In section(4) graphical representation. At the end we are present the conclusion section(5).

2. Methodologies

Consider the NPDE is,

$$\kappa(v, D_tv, D_xv, D_t^2v, ...)=0. \tag{2.1}$$

Here, $\nu = \nu(x,t)$ is a unknown function.
Suppose the travelling wave is,

$$\nu(x,t) = \nu(\kappa), \kappa = (x - \rho t), \tag{2.2}$$

putting equation Eq. 2.2 into Eq. 2.1 then,

$$O(V, V^{'}, V^{''}, V^{'''}, ...) = 0. \tag{2.3}$$

2.1. $\left(\frac{G^{'}}{G^2}\right)$*-expansion technique*

Consider the equation,

$$V(\kappa) = a_0 + \sum_{n=1}^{N} \left[a_n \left(\frac{G^{'}}{G^2}\right)^{n} + b_n \left(\frac{G^{'}}{G^2}\right)^{-n} \right]. \tag{2.4}$$

$$\left(\frac{G^{'}}{G^2}\right)^{'} = \xi_1 + \delta_1 \left(\frac{G^{'}}{G^2}\right)^{2}. \tag{2.5}$$

Where $\delta_1 \neq 0, \xi_1 \neq 1$ are integers and a_0, a_n, and b_n are unknown constants which can determined latter.
Where $n = 1, 2, 3, ...N$ The Eq. (2.5) has three cases:
Case-1 If $\xi_1 \delta_1 > 0$,

$$\left(\frac{G^{'}}{G^2}\right) = \sqrt{\frac{\xi_1}{\delta_1}} \left(\frac{A_1 \cos\sqrt{\xi_1\delta_1}\kappa + A_2 \sin\sqrt{\xi_1\delta_1}\kappa}{A_2 \cos\sqrt{\xi_1\delta_1}\kappa - A_1 \sin\sqrt{\xi_1\delta_1}\kappa}\right), \tag{2.6}$$

where A_1 and A_2 are arbitrary nonzero constants.
Case-2 If $\xi_1 \delta_1 < 0$,

$$\left(\frac{G^{'}}{G^2}\right) = -\frac{\sqrt{|\ \xi_1\delta_1\ |}}{\delta_1} \left(\frac{A_1 \sinh(2\sqrt{|\ \xi_1\delta_1\ |}\kappa) + A_1 \cosh(2\sqrt{|\ \xi_1\delta_1\ |}\kappa) + A_2}{A_1 \sinh(2\sqrt{|\ \xi_1\delta_1\ |}\kappa) + A_1 \cosh(2\sqrt{|\ \xi_1\delta_1\ |}\kappa) - A_2}\right), \tag{2.7}$$

Case-3 If $\xi_1 = 0, \delta_1 \neq 0$,

$$\left(\frac{G^{'}}{G^2}\right) = \left(-\frac{A_1}{\delta_1(A_1\kappa + A_2)}\right). \tag{2.8}$$

To obtain the three types of solution by putting the values of unknowns a_0, a_n, and b_n and the Eq. (2.6), Eq. (2.7), Eq. (2.8) into Eq. (2.4).

2.2. Exp$(-\phi(\kappa))$ *Technique*

The equation is,

$$V(\kappa) = \alpha_0 + \alpha_1 \exp(-\phi(\kappa)) + \alpha_2 \exp(-2\phi(\kappa)) + ... + \alpha_N \exp(-N\phi(\kappa)). \tag{2.9}$$

We determine the N by homogeneous balance technique,

$$\phi^{'}(\kappa) = \exp(-\phi(\kappa)) + A_1 \exp(\phi(\kappa)) + A_1. \tag{2.10}$$

Different soliton solution mentioned below,
Case-1: If $A_2^2-4A_1>0$ and $A_1\neq 0$, then

$$\phi_1(\kappa)=\ln\left(\frac{-\sqrt{A_2^2-4A_1}\tanh\left(\frac{\sqrt{A_2^2-4A_1}(\kappa+\chi)}{2}\right)-A_2}{2A_2}\right). \tag{2.11}$$

Case-2: If $A_2^2-4A_1>0$ and $A_1=0$, and $A_2\neq 0$, then

$$\phi_2(\kappa)=-\ln\left(\frac{A_2}{\cosh(A_2(\kappa+\chi))+\sinh(A_2(\kappa+\chi))-1}\right). \tag{2.12}$$

Case-3: If $A_2^2-4A_1<0$ and $A_1\neq 0$, then

$$\phi_3(\kappa)=\ln\left(\frac{\sqrt{4A_1-A_2^2}\tanh\left(\frac{\sqrt{4A_1-A_2^2}(\kappa+\chi)}{2}\right)-A_2}{2A_2}\right). \tag{2.13}$$

Case-4: If $A_2^2-4A_1=0$, $A_1\neq 0$, and $A_2\neq 0$, then

$$\phi_4(\kappa)=\ln\left(\frac{-2A_2(\kappa+\chi)+4}{A_2^2(\kappa+\chi)}\right). \tag{2.14}$$

Case-5: If $A_2^2-4A_1=0$, $A_1=0$, and $A_2=0$, then

$$\phi_5(\kappa)=\ln(\kappa+\chi). \tag{2.15}$$

Now, by subsituting Eq. (2.9) and Eq. (2.10) into left hand side of Eq. (2.3), a polynomial in $\exp(-\phi(\kappa))$ have been attained. On solving the polynomial with the help of symbolic computational software Mathematica, the exact solutions are obtained.

3. Dynamical analysis of FONLE

Consider the wave transformation is,

$$v(x,t)=V(\kappa),\qquad \kappa=x-\rho t \tag{3.1}$$

putting above relation into Eq. (1.1) then we get

$$(1-\rho)V'+c_1VV'+c_2V'''+c_3V'V''+c_4VV'''+c_5V^{(5)}=0. \tag{3.2}$$

Integrating Eq. (3.2) once w.r.t. κ

$$\frac{1}{2}(c_3-c_4)(V')^2+c_4VV''+c_2V''+\frac{1}{2}c_1(V)^2+(1-\rho)V+c_5V^{(4)}=0. \tag{3.3}$$

Where the constant of integration is zero.

3.1. $\left(\frac{G'}{G^2}\right)$-expansion technique and its Application

By homogeneous balance technique we get $N = 2$.
Inserting $N = 2$ in Eq. (2.4), then it become

$$V(\kappa) = a_0 + a_1\left(\frac{G'}{G^2}\right) + b_1\left(\frac{G'}{G^2}\right)^{-1} + a_2\left(\frac{G'}{G^2}\right)^{2} + b_2\left(\frac{G'}{G^2}\right)^{-2}. \tag{3.4}$$

Substituting Eq. (3.4) into Eq. (3.3) then we get following set of solution,
Set 1

$$a_0 = \mp\frac{5\iota b_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}, a_1 = -\frac{b_1\delta_1}{\xi_1}, a_2 = \mp\frac{\iota b_1\delta_1^{\frac{3}{2}}}{4\xi_1^{\frac{3}{2}}}, b_1 = b_1, b_2 = \mp\frac{\iota b_1\sqrt{\xi_1}}{4\sqrt{\delta_1}}, \rho = 9216c_5\delta_1^2\xi_1^2 + 1,$$
$$c_1 = \pm\frac{4608\iota c_5\delta_1^{\frac{3}{2}}\xi_1^{\frac{5}{2}}}{b_1}, c_2 = -560c_5\delta_1\xi_1, c_3 = \mp\frac{624\iota c_5\sqrt{\delta_1}\xi_1^{\frac{3}{2}}}{b_1}, c_4 = \pm\frac{192\iota c_5\sqrt{\delta_1}\xi_1^{\frac{3}{2}}}{b_1}. \tag{3.5}$$

Case 1 If $\xi_1\delta_1 > 0$,

$$\begin{aligned}V(\kappa) = {} & \left(\mp\frac{5\iota b_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}\right) + \left(-\frac{b_1\delta_1}{\xi_1}\right)\left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right) \\ & + b_1\left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right)^{-1} + \left(\mp\frac{\iota b_1\delta_1^{\frac{3}{2}}}{4\xi_1^{\frac{3}{2}}}\right) \\ & \left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right)^{2} + \left(\mp\frac{\iota b_1\sqrt{\xi_1}}{4\sqrt{\delta_1}}\right) \\ & \left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right)^{-2}.\end{aligned} \tag{3.6}$$

Where $\kappa = x - \rho t$
case 2 If $\xi_1\delta_1 < 0$.

$$\begin{aligned}V(\kappa) = {} & \left(\mp\frac{5\iota b_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}\right) + \left(-\frac{b_1\delta_1}{\xi_1}\right)\left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right) \\ & + b_1\left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right)^{-1} + \left(\mp\frac{\iota b_1\delta_1^{\frac{3}{2}}}{4\xi_1^{\frac{3}{2}}}\right) \\ & \left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right)^{2} + \left(\mp\frac{\iota b_1\sqrt{\xi_1}}{4\sqrt{\delta_1}}\right) \\ & \left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right)^{-2}.\end{aligned} \tag{3.7}$$

Case-3 If $\xi_1 = 0, \delta_1 \neq 0$

$$V(\kappa) = \left(\mp\frac{5\iota b_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}\right) + \left(-\frac{b_1\delta_1}{\xi_1}\right)\left(-\frac{A_1}{\delta_1(A_1\kappa + A_2)}\right) + b_1\left(-\frac{A_1}{\delta_1(A_1\kappa + A_2)}\right)^{-1} + \left(\mp\frac{\iota b_1\delta_1^{\frac{3}{2}}}{4\xi_1^{\frac{3}{2}}}\right)\left(-\frac{A_1}{\delta_1(A_1\kappa + A_2)}\right)^2 + \left(\mp\frac{\iota b_1\sqrt{\xi_1}}{4\sqrt{\delta_1}}\right)\left(-\frac{A_1}{\delta_1(A_1\kappa + A_2)}\right)^{-2}. \tag{3.8}$$

Set 2

$$a_0 = \pm\frac{3\iota a_1\sqrt{\xi_1}}{2\sqrt{\delta_1}}, a_1 = a_1, a_2 = \pm\frac{\iota a_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}, b_1 = 0, b_2 = 0, \rho = 576c_5\delta_1^2\xi_1^2 + 1, c_1 = \mp\frac{576\iota c_5\delta_1^{\frac{5}{2}}\xi_1^{\frac{3}{2}}}{a_1},$$
$$c_2 = -140c_5\delta_1\xi_1, c_3 = \pm\frac{312\iota c_5\delta_1^{\frac{3}{2}}\sqrt{\xi_1}}{a_1}, c_4 = \mp\frac{96\iota c_5\delta_1^{\frac{3}{2}}\sqrt{\xi_1}}{a_1}. \tag{3.9}$$

Case 1 If $\xi_1\delta_1 > 0$,

$$V(\kappa) = \left(\pm\frac{3\iota a_1\sqrt{\xi_1}}{2\sqrt{\delta_1}}\right) + a_1\left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right) + \left(\pm\frac{\iota a_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}\right)\left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right)^2. \tag{3.10}$$

case 2 If $\xi_1\delta_1 < 0$.

$$V(\kappa) = \left(\pm\frac{3\iota a_1\sqrt{\xi_1}}{2\sqrt{\delta_1}}\right) + a_1\left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right) + \left(\pm\frac{\iota a_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}\right)\left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right)^2. \tag{3.11}$$

Case-3 If $\xi_1 = 0, \delta_1 \neq 0$

$$V(\kappa) = \left(\pm\frac{3\iota a_1\sqrt{\xi_1}}{2\sqrt{\delta_1}}\right) + a_1\left(-\frac{A_1}{\delta_1(A_1\kappa + B_1)}\right) + \left(\pm\frac{\iota a_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}\right)\left(-\frac{A_1}{\delta_1(A_1\kappa + A_2)}\right)^2. \tag{3.12}$$

Set 3

$$a_0 = \frac{2b_2\delta_1}{\xi_1}, a_1 = 0, a_2 = \frac{b_2\delta_1^2}{\xi_1^2}, b_1 = 0, b_2 = b_2, \rho = \frac{3\xi_1 + 4b_2c_1\delta_1}{3\xi_1}, c_2 = \frac{-b_2c_1 + 192c_5\delta_1\xi_1^3}{12\xi_1^2},$$
$$c_3 = -\frac{36c_5\xi_1^2}{b_2}, c_4 = -\frac{12c_5\xi_1^2}{b_2} \tag{3.13}$$

Case 1 If $\xi_1\delta_1 > 0$,

$$\begin{aligned} V(\kappa) &= \left(\frac{2b_2\delta_1}{\xi_1}\right) + \left(\frac{b_2\delta_1^2}{\xi_1^2}\right)\left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right)^2 \\ &+ b_2\left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right)^{-2}. \end{aligned} \tag{3.14}$$

Case 2 If $\xi_1\delta_1 < 0$,

$$\begin{aligned} V(\kappa) &= \left(\frac{2b_2\delta_1}{\xi_1}\right) + \left(\frac{b_2\delta_1^2}{\xi_1^2}\right)\left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right)^2 \\ &+ b_2\left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right)^{-2}. \end{aligned} \tag{3.15}$$

Case-3 If $\xi_1 = 0, \delta_1 \neq 0$

$$V(\kappa) = \left(\frac{2b_2\delta_1}{\xi_1}\right) + \left(\frac{b_2\delta_1^2}{\xi_1^2}\right)\left(-\frac{A_1}{\delta_1(A_1\kappa + A_2)}\right)^2 + b_2\left(-\frac{A}{\delta_1(A_1\kappa + A_2)}\right)^{-2}. \tag{3.16}$$

Set 4

$$\begin{aligned} &a_0 = \frac{\left(\frac{1}{6} \pm i\right) a_1\sqrt{\xi_1}}{\sqrt{\delta_1}}, a_2 = \frac{a_1\sqrt{\delta_1}}{6\sqrt{\xi_1}}, b_1 = 0, b_2 = 0, \rho = 1 + 1152ic_5\delta_1^2\xi_1^2, c_1 = \frac{1152c_5\delta_1^{5/2}\xi_1^{3/2}}{a_1}, \\ &c_2 = (4 - 288i)c_5\delta_1\xi_1, c_3 = -\frac{936c_5\delta_1^{3/2}\sqrt{\xi_1}}{a_1}, c_4 = \frac{288c_5\delta_1^{3/2}\sqrt{\xi_1}}{a_1}. \end{aligned} \tag{3.17}$$

Case 1 If $\xi_1\delta_1 > 0$,

$$\begin{aligned} V(\kappa) &= \left(\frac{\left(\frac{1}{6} \pm i\right) a_1\sqrt{\xi_1}}{\sqrt{\delta_1}}\right) + a_1\left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right) \\ &+ \left(\frac{a_1\sqrt{\delta_1}}{6\sqrt{\xi_1}}\right)\left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right)^2. \end{aligned} \tag{3.18}$$

Case 2 If $\xi_1\delta_1 < 0$,

$$\begin{aligned} V(\kappa) &= \left(\frac{\left(\frac{1}{6} \pm i\right) a_1\sqrt{\xi_1}}{\sqrt{\delta_1}}\right) + a_1\left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right) \\ &+ \left(\frac{a_1\sqrt{\delta_1}}{6\sqrt{\xi_1}}\right)\left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right)^2. \end{aligned} \tag{3.19}$$

Case-3 If $\xi_1 = 0, \delta_1 \neq 0$

$$V(\kappa) = \left(\frac{\left(\frac{1}{6} \pm i\right) a_1\sqrt{\xi_1}}{\sqrt{\delta_1}}\right) + a_1\left(-\frac{A_1}{\delta_1(A_1\kappa + A_2)}\right) + \left(\frac{a_1\sqrt{\delta_1}}{6\sqrt{\xi_1}}\right)\left(-\frac{A_1}{\delta_1(A_1\kappa + A_2)}\right)^2. \tag{3.20}$$

Set 5

$$a_0 = -\frac{ia_1\sqrt{\xi_1}}{2\sqrt{\delta_1}}, a_2 = \frac{ia_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}, b_1 = 0, b_2 = 0, \rho = 1 - 576c_5\delta_1^2\xi_1^2, c_1 = -\frac{576ic_5\delta_1^{5/2}\xi_1^{3/2}}{a_1},$$
$$c_2 = 52c_5\delta_1\xi_1, c_3 = \frac{312ic_5\delta_1^{3/2}\sqrt{\xi_1}}{a_1}, c_4 = -\frac{96ic_5\delta_1^{3/2}\sqrt{\xi_1}}{a_1} \tag{3.21}$$

Case 1 If $\xi_1\delta_1 > 0$,

$$\begin{aligned} V(\kappa) &= \left(-\frac{ia_1\sqrt{\xi_1}}{2\sqrt{\delta_1}}\right) + a_1\left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right) \\ &+ \left(\frac{ia_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}\right)\left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right)^2. \end{aligned} \tag{3.22}$$

Case 2 If $\xi_1\delta_1 < 0$,

$$\begin{aligned} V(\kappa) &= \left(-\frac{ia_1\sqrt{\xi_1}}{2\sqrt{\delta_1}}\right) + a_1\left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right) \\ &+ \left(\frac{ia_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}\right)\left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right)^2. \end{aligned} \tag{3.23}$$

Case-3 If $\xi_1 = 0, \delta_1 \neq 0$

$$V(\kappa) = \left(-\frac{ia_1\sqrt{\xi_1}}{2\sqrt{\delta_1}}\right) + a_1\left(-\frac{A_1}{\delta_1(A_1\kappa + A_2)}\right) + \left(\frac{ia_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}\right)\left(-\frac{A_1}{\delta_1(A_1\kappa + A_2)}\right)^2. \tag{3.24}$$

Set 6

$$a_0 = -\frac{ib_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}, a_1 = 0, a_2 = 0, b_2 = \frac{ib_1\sqrt{\xi_1}}{2\sqrt{\delta_1}}, \rho = 1 - 576c_5\delta_1^2\xi_1^2, c_1 = -\frac{576ic_5\delta_1^{3/2}\xi_1^{5/2}}{b_1},$$
$$c_2 = 52c_5\delta_1\xi_1, c_3 = \frac{312ic_5\sqrt{\delta_1}\xi_1^{3/2}}{b_1}, c_4 = -\frac{96ic_5\sqrt{\delta_1}\xi_1^{3/2}}{b_1} \tag{3.25}$$

Case 1 If $\xi_1\delta_1 > 0$,

$$\begin{aligned} V(\kappa) &= \left(-\frac{ib_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}\right) + b_1\left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right)^{-1} \\ &+ \left(\frac{ib_1\sqrt{\xi_1}}{2\sqrt{\delta_1}}\right)\left(\sqrt{\frac{\xi_1}{\delta_1}}\left(\frac{A_1\cos\sqrt{\xi_1\delta_1}\kappa + A_2\sin\sqrt{\xi_1\delta_1}\kappa}{A_2\cos\sqrt{\xi_1\delta_1}\kappa - A_1\sin\sqrt{\xi_1\delta_1}\kappa}\right)\right)^{-2}. \end{aligned} \tag{3.26}$$

Case 2 If $\xi_1\delta_1 < 0$,

$$V(\kappa) = \left(-\frac{ib_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}\right) + b_1\left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right)^{-1}$$
$$+\left(\frac{ib_1\sqrt{\xi_1}}{2\sqrt{\delta_1}}\right)\left(-\frac{\sqrt{|\xi_1\delta_1|}}{\delta_1}\left(\frac{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_2}{A_1\sinh(2\sqrt{|\xi_1\delta_1|}\kappa) + A_1\cosh(2\sqrt{|\xi_1\delta_1|}\kappa) - A_2}\right)\right)^{-2}. \tag{3.27}$$

Case-3 If $\xi_1 = 0, \delta_1 \neq 0$

$$V(\kappa) = \left(-\frac{ib_1\sqrt{\delta_1}}{2\sqrt{\xi_1}}\right) + b_1\left(-\frac{A_1}{\delta_1(A_1\kappa + A_2)}\right)^{-1} + \left(\frac{ib_1\sqrt{\xi_1}}{2\sqrt{\delta_1}}\right)\left(-\frac{A_1}{\delta_1(A_1\kappa + A_2)}\right)^{-2}. \tag{3.28}$$

3.2. *The* $\exp(-\phi(\kappa))$ *method and its Application*

By homogeneous balance technique we attain $N = 2$.
Substituting $N = 2$ in Eq. (2.9), then it become

$$V(\kappa) = \alpha_0 + \alpha_1\exp(-\phi(\kappa)) + \alpha_2\exp(-2\phi(\kappa)). \tag{3.29}$$

Putting (3.29) into (3.3) then we attain the following set of solution,
Set 1

$$\alpha_0 = -\frac{12A_1c_5}{c_4}, \alpha_1 = -\frac{12A_2c_5}{c_4}, c_1 = \frac{-4A_1c_4c_5 + A_2^2c_4c_5 + c_2c_4}{c_5}, \alpha_2 = -\frac{12c_5}{c_4}, c_3 = 3c_4,$$
$$\rho = 16A_1^2c_5 - 8A_1A_2^2c_5 - 4A_1c_2 + A_2^4c_5 + A_2^2c_2 + 1 \tag{3.30}$$

Case-1:If $A_2^2 - 4A_1 > 0$ and $A_1 \neq 0$, then

$$V(\kappa) = \left(-\frac{12A_1c_5}{c_4}\right) + \left(-\frac{12A_2c_5}{c_4}\right)\left(\frac{-\sqrt{A_2^2-4A_1}\tanh\left(\frac{\sqrt{A_2^2-4A_1}(\kappa+\chi)}{2}\right) - A_2}{2A_2}\right)^{-1} + \left(-\frac{12c_5}{c_4}\right)$$
$$\left(\frac{-\sqrt{A_2^2-4A_1}\tanh\left(\frac{\sqrt{A_2^2-4A_1}(\kappa+\chi)}{2}\right) - A_2}{2A_2}\right)^{-2}. \tag{3.31}$$

Case-2:If $A_2^2 - 4A_1 > 0$ and $A_1 = 0$, and $A_2 \neq 0$,

$$V(\kappa) = \left(-\frac{12A_1c_5}{c_4}\right) + \left(-\frac{12A_2c_5}{c_4}\right)\left(\frac{A_2}{\cosh(A_2(\kappa+\chi)) + \sinh(A_2(\kappa+\chi)) - 1}\right) + \left(-\frac{12c_5}{c_4}\right)$$
$$\left(\frac{A_2}{\cosh(A_2(\kappa+\chi)) + \sinh(A_2(\kappa+\chi)) - 1}\right)^2. \tag{3.32}$$

Case-3:If $A_2^2-4A_1<0$ and $A_1\neq 0$,

$$V(\kappa)=\left(-\frac{12A_1c_5}{c_4}\right)+\left(\frac{12A_2c_5}{c_4}\right)\left(\frac{-\sqrt{4A_1-A_2^2}\tanh\left(\frac{\sqrt{4A_1-A_2^2}(\kappa+\chi)}{2}\right)-A_2}{2A_2}\right)^{-1}+\left(-\frac{12c_5}{c_4}\right)\left(\frac{\sqrt{4A_1-A_2^2}\tanh\left(\frac{\sqrt{4A_1-A_2^2}(\kappa+\chi)}{2}\right)-A_2}{2A_2}\right)^{-2}. \tag{3.33}$$

Case-4:If $A_2^2-4A_1=0$, $A_1\neq 0$, and $B_1\neq 0$,

$$V(\kappa)=\left(-\frac{12A_1c_5}{c_4}\right)+\left(-\frac{12A_2c_5}{c_4}\right)\left(\frac{-2A_2(\kappa+\chi)+4}{A_2^2(\kappa+\chi)}\right)^{-1}+\left(-\frac{12c_5}{c_4}\right)\left(\frac{-2A_2(\kappa+\chi)+4}{A_2^2(\kappa+\chi)}\right)^{-2}. \tag{3.34}$$

Case-5:If $A_2^2-4A_1=0$, $A_1=0$, and $A_2=0$,

$$V(\kappa)=\left(-\frac{12A_1c_5}{c_4}\right)+\left(-\frac{12A_2c_5}{c_4}\right)(\kappa+\chi)^{-1}+\left(-\frac{12c_5}{c_4}\right)(\kappa+\chi)^{-2}. \tag{3.35}$$

Set 2

$$\alpha_0=\frac{48A_1c_5}{c_4},\alpha_1=\frac{48A_2c_5}{c_4},c_1=-\frac{c_4\left(16A_1c_5-4A_2^2c_5+c_2\right)}{4c_5},\alpha_2=\frac{48c_5}{c_4},c_3=-\frac{1}{4}\left(13c_4\right),$$
$$\rho=16A_2^2c_5-8A_1A_2^2c_5-4A_1c_2+A_2^4c_5+A_2^2c_2+1 \tag{3.36}$$

Case-1:If $A_2^2-4A_1>0$ and $A_1\neq 0$, then

$$V(\kappa)=\left(\frac{48A_1c_5}{c_4}\right)+\left(\frac{48A_2c_5}{c_4}\right)\left(\frac{-\sqrt{A_2^2-4A_1}\tanh\left(\frac{\sqrt{A_2^2-4A_1}(\kappa+\chi)}{2}\right)-A_2}{2A_2}\right)^{-1}+\left(\frac{48c_5}{c_4}\right)\left(\frac{-\sqrt{A_2^2-4A_1}\tanh\left(\frac{\sqrt{A_2^2-4A_1}(\kappa+\chi)}{2}\right)-A_2}{2A_2}\right)^{-2}. \tag{3.37}$$

Case-2:If $A_2^2-4A_1>0$ and $A_1=0$, and $A_2\neq 0$,

$$V(\kappa)=\left(\frac{48A_1c_5}{c_4}\right)+\left(\frac{48A_2c_5}{c_4}\right)\left(\frac{A_2}{\cosh(A_2(\kappa+\chi))+\sinh(A_2(\kappa+\chi))-1}\right)+\left(\frac{48c_5}{c_4}\right)\left(\frac{A_2}{\cosh(A_2(\kappa+\chi))+\sinh(A_2(\kappa+\chi))-1}\right)^{2}. \tag{3.38}$$

Case-3:If $A_2^2-4A_1<0$ and $A_1\neq 0$,

$$V(\kappa)=\left(\frac{48A_1c_5}{c_4}\right)+\left(\frac{48A_2c_5}{c_4}\right)\left(\frac{\sqrt{4A_1-A_2^2}\tanh\left(\frac{\sqrt{4A_1-A_2^2}(\kappa+\chi)}{2}\right)-A_2}{2A_2}\right)^{-1}+\left(\frac{48c_5}{c_4}\right)\left(\frac{\sqrt{4A_1-A_2^2}\tanh\left(\frac{\sqrt{4A_1-A_2^2}(\kappa+\chi)}{2}\right)-A_2}{2A_2}\right)^{-2}. \tag{3.39}$$

Case-4:If $A_2^2-4A_1=0$, $A_1\neq 0$, and $A_2\neq 0$,

$$V(\kappa)=\left(\frac{48A_1c_5}{c_4}\right)+\left(\frac{48A_2c_5}{c_4}\right)\left(\frac{-2A_2(\kappa+\chi)+4}{A_2^2(\kappa+\chi)}\right)^{-1}+\left(\frac{48c_5}{c_4}\right)\left(\frac{-2A_2(\kappa+\chi)+4}{A_2^2(\kappa+\chi)}\right)^{-2}. \tag{3.40}$$

Case-5:If $A_2^2-4A_1=0$, $A_1=0$, and $A_2=0$,

$$V(\kappa)=\left(\frac{48A_1c_5}{c_4}\right)+\left(\frac{48A_2c_5}{c_4}\right)(\kappa+\chi)^{-1}+\left(\frac{48c_5}{c_4}\right)(\kappa+\chi)^{-2}. \tag{3.41}$$

Set 3

$$\alpha_0=-\frac{2c_5\left(2A_1+A_2^2\right)}{c_4},\alpha_1=-\frac{12A_2c_5}{c_4},c_1=\frac{c_4\left(4A_1c_5+A_2^2\left(-c_5\right)+c_2\right)}{c_5},\alpha_2=-\frac{12c_5}{c_4},$$
$$c_3=3c_4,\rho=16A_1^2c_5-8A_1A_2^2c_5+4A_1c_2+A_2^4c-A_2^2c_2+1 \tag{3.42}$$

Case-1:If $A_2^2-4A_1>0$ and $A_1\neq 0$, then

$$V(\kappa)=\left(-\frac{2c_5\left(2A_1+A_2^2\right)}{c_4}\right)+\left(-\frac{12A_2c_5}{c_4}\right)\left(\frac{-\sqrt{A_2^2-4A_1}\tanh\left(\frac{\sqrt{A_2^2-4A_1}(\kappa+\chi)}{2}\right)-A_2}{2A_2}\right)^{-1}+\left(-\frac{12c_5}{c_4}\right)\left(\frac{-\sqrt{A_2^2-4A_1}\tanh\left(\frac{\sqrt{A_2^2-4A_1}(\kappa+\chi)}{2}\right)-A_2}{2A_2}\right)^{-2}. \tag{3.43}$$

Case-2:If $A_2^2-4A_1>0$ and $A_1=0$, and $A_2\neq 0$,

$$V(\kappa)=\left(-\frac{2c_5\left(2A_1+A_2^2\right)}{c_4}\right)+\left(-\frac{12A_2c_5}{c_4}\right)\left(\frac{A_2}{\cosh(A_2(\kappa+\chi))+\sinh(A_2(\kappa+\chi))-1}\right)+\left(-\frac{12c_5}{c_4}\right)\left(\frac{A_2}{\cosh(A_2(\kappa+\chi))+\sinh(A_2(\kappa+\chi))-1}\right)^2. \tag{3.44}$$

Case-3:If $A_2^2 - 4A_1 < 0$ and $A_1 \neq 0$,

$$V(\kappa) = \left(-\frac{2c_5\left(2A_1 + A_2^2\right)}{c_4}\right) + \left(-\frac{12A_2c_5}{c_4}\right)\left(\frac{\sqrt{4A_1 - A_2^2}\tanh\left(\frac{\sqrt{4A_1 - A_2^2}(\kappa+\chi)}{2}\right) - A_2}{2A_2}\right)^{-1} + \left(-\frac{12c_5}{c_4}\right)\left(\frac{\sqrt{4A_1 - A_2^2}\tanh\left(\frac{\sqrt{4A_1 - A_2^2}(\kappa+\chi)}{2}\right) - A_2}{2A_2}\right)^{-2}. \tag{3.45}$$

Case-4:If $A_2^2 - 4A_1 = 0$, $A_1 \neq 0$, and $A_2 \neq 0$,

$$V(\kappa) = \left(-\frac{2c_5\left(2A_1 + A_2^2\right)}{c_4}\right) + \left(-\frac{12A_2c_5}{c_4}\right)\left(\frac{-2A_2(\kappa+\chi) + 4}{A_2^2(\kappa+\chi)}\right)^{-1} + \left(-\frac{12c_5}{c_4}\right)\left(\frac{-2A_2(\kappa+\chi) + 4}{A_2^2(\kappa+\chi)}\right)^{-2}. \tag{3.46}$$

Case-5:If $A_2^2 - 4A_1 = 0$, $A_1 = 0$, and $A_2 = 0$,

$$V(\kappa) = \left(-\frac{2c_5\left(2A_1 + A_2^2\right)}{c_4}\right) + \left(-\frac{12A_2c_5}{c_4}\right)(\kappa+\chi)^{-1} + \left(-\frac{12c_5}{c_4}\right)(\kappa+\chi)^{-2}. \tag{3.47}$$

4. Graphically Discussion

In this section, the graphical visualization of FONLE equation have been discussed. The physical nature of the nonlinear model is illustrated by setting suitable values to the arbitrary constants with the help of Mathematica. The Fig.(1) shows the singular-periodic evolution; moreover, Fig.(2) represents the kink-soliton, and Fig.(3) is again a singular-periodic wave. Fig.(4) represents the stable bright soliton, and Fig.(5) and Fig.(6) are singular-bright type solitary waves and singular-kink solitary waves, respectively. "Singular-periodic" usually refers to solutions that show both periodic behaviour and singularities in nonlinear waves or wave equations. These solutions frequently occur in nonlinear systems, where intriguing dynamical phenomena are produced by the interaction between periodicity and singular behaviour. Kink soliton, alternatively called topological soliton or just kink, is a kind of solitary wave that appears in some nonlinear field theories, especially in condensed matter physics and classical field theory. A smooth, localised disturbance in a field that interpolates between two different equilibrium states is what defines a kink. They frequently arise in systems whose symmetries spontaneously break, where the field can assume many values in various spatial locations. One kind of solitary wave that does not change form throughout propagation is the bright soliton, which keeps its amplitude and shape constant. Usually, it is related to nonlinear wave equations, especially when nonlinear optics and Bose-Einstein condensates are involved. A balance between dispersion, which tends to spread out the wave, and nonlinearity, which tends to focus it, produces bright solitons. Singular bright soliton might be used to describe soliton solutions that show solitary behaviour along with localised brilliant characteristics, including solution discontinuities or sites of divergence. This might correspond to a particular kind of nonlinear wave solution in which the nonlinearity produces a localised amplitude or energy concentration, possibly together with singularities in the solution profile.

Figure 1:
3D, 2D and contour singular-periodic solitary wave shape of Eq. (3.6) when $\xi_1 = 0.1, A_1 = 0.5, A_2 = 0.02, b_1 = 0.2, \delta_1 = 0.5, c_5 = 0.1$

Figure 2:
2D and 3D kink solitary wave shape of Eq. (3.11) when $\xi_1 = 0.02, A_1 = 4, A_2 = -5, a_1 = 0.2, b_1 = 0.05, \delta_1 = 0.5, c_5 = 4$

Figure 3:
2D and 3D singular-periodic solitary wave shape of Eq. (3.14) when $\xi_1 = 0.5, A_1 = 1, A_2 = 2, c_1 = 0.3, b_2 = 0.2, \delta_1 = 0.2, c_5 = -0.2$

Figure 4:
2D and 3D bright solitary wave shape of Eq. (3.37) when $\chi = -3, A_1 = 0.01, c_2 = 0.5, A_2 = 1, c_4 = 0.02, c_5 = -0.05$

Figure 5:
2D and 3D singular-bright solitary wave shape of Eq. (3.38) when $\chi = 1.5, A_1 = 0, c_2 = -0.8, A_2 = 2, c_4 = 0.05, c_5 = 0.2$

Figure 6:
2D and 3D singular-kink solitary wave shape of Eq. (3.39) when $\chi = -0.05, A_1 = 1.5, c_2 = -0.8, A_2 = 0.5, c_4 = -0.05, c_5 = -0.2$

5. Conclusion

In this article, exact solutions for the FONLE equation are symbolically attained by using $\left(\frac{G'}{G^2}\right)$-expansion method and the $\exp(-\phi(\kappa))$ method. The singular-kink, singular-periodic and bright solitary waves are obtained by using the given methods. These solutions help us to acknowledge complex physical phenomena in applied mathematics and physics. These methods can be utilized to obtain localized wave solutions for different nonlinear water wave equations in mathematics, engineering and physics. The outcomes of the research could contribute to a better understanding of the physical relevance of the model under investigation as well as other nonlinear models that are often used in the research of optical fibers, quantum field theory, and other related topics. At the end, we have successfully analyze the physical phenomena of FONEL equation through the 2D and 3D graphs.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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